Team Selection Test Selection Test Solutions

1. Let $\leftarrow$ denote the left arrow key on a standard keyboard. If one opens a text editor and types the keys "ab$\leftarrow$cd$\leftarrow\leftarrow$e$\leftarrow\leftarrow$f", the result is "faecdb". We say that a string B is *reachable* from a string A if it is possible to insert some amount of $\leftarrow$'s into A, such that typing the resulting characters produces B. So, our example shows that "faecdb" is reachable from "abcdef".

   Prove that for any two strings A and B, A is reachable from B if and only if B is reachable from A.

   Solution 1. Proceed by strong induction on the length of the strings.. The base case, the empty string, is trivial.

   For the inductive step, suppose B is reachable from A. Without loss of generality, suppose all of the characters in A are distinct. Let a be the first character in A. Then B is of the form XaY, where X and Y are (possibly empty) strings. Then, it is not difficult to check that A must be of the form $aY'X'$, where Y' can reach Y and X' can reach X. By the inductive hypothesis, Y can reach Y' and X can reach X'.

   Therefore, we can reach A from B, as follows:

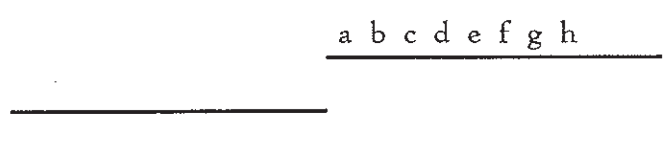
   - Type a.
   - Insert $\leftarrow$ into Y', and type it to produce Y.
   - Press $\leftarrow$ until the cursor is at the start of the string.
   - Insert $\leftarrow$ into X', and type it to produce X.

   This produces B, as desired.

   Solution 2. We give a sketch. Without loss of generality, let A be the string $123\cdots n$ (each integer is a character). Then, B is a permutation of $1, 2, \ldots, n$. Call a permutation σ *reachable* if the corresponding string B is reachable from A. One can check the following two facts:

   - σ is reachable if and only if it is *213-avoiding*, that is, there do not exist $i < j < k$ such that $\sigma(j) < \sigma(i) < \sigma(k)$. In other words, the permutation avoids three (not necessarily distinct) numbers in the order medium, low, high.
   - σ is 213-avoiding if and only if σ^{-1} is 213-avoiding.

   These two facts imply the problem statement.



   Solution 3. We can model the situation using three conveyor belts (see above), as follows:

   - The top conveyor belt initially contains the string A.

- The middle conveyor belt represents the part of the string before the cursor.
- The bottom conveyor belt represents the part of the string after the cursor.

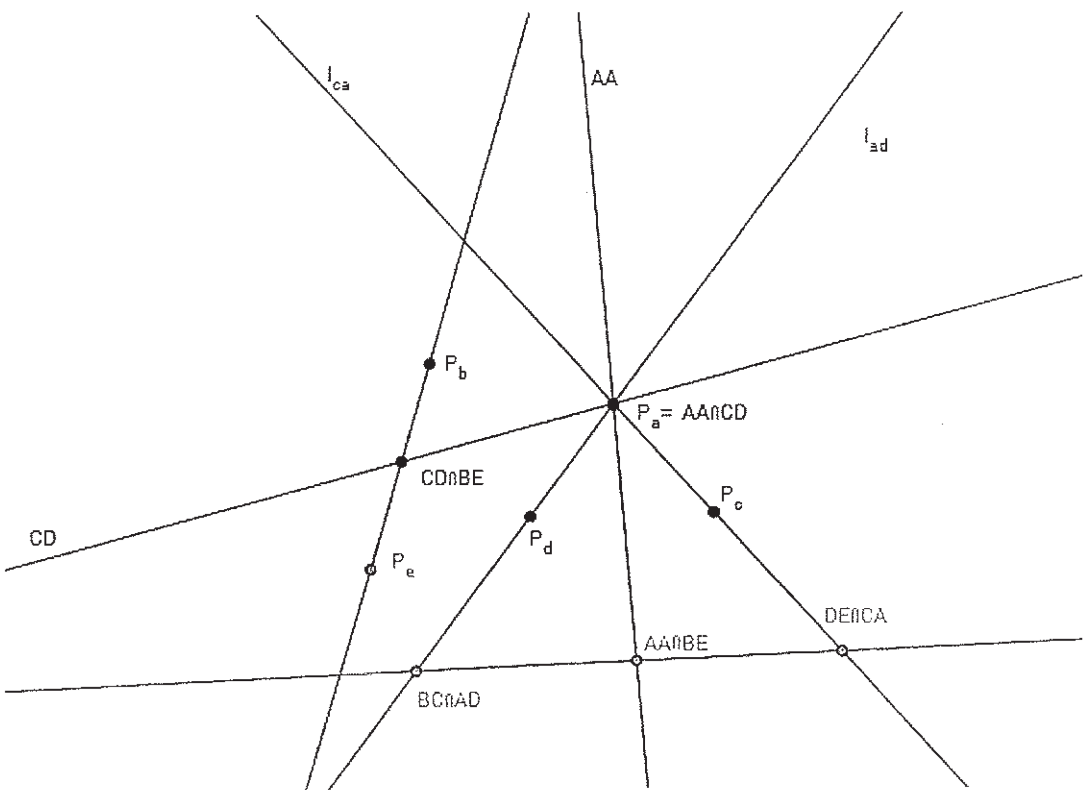
Whenever we type the next letter in A, we roll the top conveyor belt one space left. The leftmost letter falls onto the middle conveyor belt, to the right of any letters that are already there. Whenever we type $\leftarrow$, we roll the middle conveyor belt one space right. The rightmost letter falls onto the bottom conveyor belt, to the left of any letters that are already there. At the end of this procedure, the bottom conveyor belt will hold the final string B. This procedure represents a way to reach B from A.

Now, we must reach A from B. To do this, switch the top and bottom conveyor belts and then reverse time.

2. Consider a convex pentagon circumscribed about a circle. We name the lines that connect vertices of the pentagon with the opposite points of tangency with the circle *gergonnians*.

   (a) Prove that if four gergonnians are concurrent, then all five of them are concurrent.

   (b) Prove that if there is a triple of gergonnians that are concurrent, then you can find another triple of gergonnians that are concurrent.



Solution 1. Let A, B, C, D, E be the tangency points, and A', B', C', D', E' be the vertices opposite the tangency points (so $A' = CC \cap DD$, $B' = DD \cap EE$, etc.). Throughout this

solution we work in the projective plane, and denote by XX the tangent to the circle at a point X.

It suffices to show that the gergonians AA', BB', EE' concur if and only if the gergonians AA', CC', DD' concur. Indeed, by symmetry this immediately implies (b) (whether or not the three gergonians in question are all adjacent); to get (a), if we suppose AA', BB', CC', DD' concur at some (unique) point P, then AA', BB', EE' concur at $AA' \cap BB' = AA' \cap BB' \cap CC' \cap DD' = P$, so all five gergonians concur at P.

This is possible to show by repeated applications of Brianchon's Theorem, but instead we begin by "taking the projective dual." Note that A' is the pole of DC, and A is the pole of AA; by a standard fact of poles and polars (sometimes called La Hire's Theorem), AA' is the polar of the point $P_a := AA \cap DC$. Similarly define P_b, P_c, P_d, P_e, and observe (again by pole-polar duality) that the gergonigans XX', YY', ZZ' concur if and only if their poles P_x, P_y, P_z are collinear.

In our new formulation, we need to show that P_a, P_b, P_e are collinear if and only if P_a, P_c, P_d are collinear. Let ℓ_{xy} denote the line formed by P_x, P_y. (It's easy to see that the P_x are pairwise distinct, since the gergonians XX' are pairwise distinct.) Pascal's Theorem on hexagon $AABCDD$ yields $P_a, P_d, BC \cap AD$ are collinear on line ℓ_{ad}, or equivalently that ℓ_{ad}, BC, AD concur at a point Q_{ad}. (As a slightly cumbersome detail, note that the points $P_a, P_d, BC \cap AD$ are pairwise distinct, and the lines ℓ_{ad}, BC, AD are pairwise distinct.)Similarly define Q_{db}, Q_{be}, Q_{ec}, Q_{ca}.

Now we chase equivalences. First, P_a, P_b, P_e are collinear if and only if $P_a \in \ell_{be}$ if and only if AA, CD, ℓ_{be} are concurrent; in turn this is true if and only if the unique intersection $Q_{be} = CD \cap \ell_{be}$ lies on AA. But $Q_{be} = CD \cap BE$ as well, so P_a, P_b, P_e *are collinear if and only if the pairwise distinct lines* AA, CD, BE *concur.*

On the other hand, P_a, P_c, P_d are collinear if and only if $P_aP_d = \ell_{ad} = P_a(BC \cap AD)$ and $P_cP_a = \ell_{ca} = P_a(DE \cap CA)$ coincide[2] if and only if the three pairwise distinct points $P_a, BC \cap AD, DE \cap CA$ are collinear.

Finally, Pascal's Theorem on $BCAADE$ yields $BC \cap AD$, $CA \cap DE$, $AA \cap BE$ collinear. But we know P_a, P_c, P_d are collinear if and only if $P_a = AA \cap CD$ lies on the line through (the distinct points) $BC \cap AD, DE \cap CA$. As just shown, this line intersects (the distinct line) AA in a single point $AA \cap BE$, so P_a, P_c, P_d *are collinear if and only if the pairwise distinct lines* AA, CD, BE *concur.*

Hence P_a, P_b, P_e are collinear if and only if the pairwise distinct lines AA, CD, BE concur, which is true if and only if P_a, P_c, P_d are collinear, so we're done.

3. Find all polynomial functions $P(x)$ with real coefficients that satisfy

$$P(x\sqrt{2}) = P(x + \sqrt{1 - x^2})$$

for all real x with $|x| \leq 1$.

[2]As mentioned earlier, $P_a, P_d, BC \cap AD$ are pairwise distinct, as are $P_a, P_c, P_D E \cap CA$.

Solution 1. Let t be a real number with $0 < t < \sqrt{2}$. Taking $x = t/\sqrt{2}$ and $x = \sqrt{(2-t^2)/2}$ in the given yields:

$$P(t) = P\left(t/\sqrt{2} + \sqrt{1 - t^2/2}\right) = P\left(\sqrt{2-t^2}\right)$$

We now decompose P into its odd and even parts. Let $P(x) = Q(x^2) + x \cdot R(x^2)$. If R is not the zero polynomial, plugging in $x = \sqrt{2-t^2}$ into the above and rearranging yields:

$$\begin{aligned}\sqrt{2-t^2} &= \frac{P\left(\sqrt{2-t^2}\right) - Q(2-t^2)}{R(2-t^2)} \\ &= \frac{P(t) - Q(2-t^2)}{R(2-t^2)}\end{aligned}$$

But this implies that we can express $\sqrt{2-t^2}$ as a rational function in t for all $t \in (0,1)$ with $R(t) \neq 0$, a contradiction. So $R(x) = 0$ for all x and $P(x) = Q(x^2)$.

We now note that, for all r with $0 < r < 1$, we have $Q(r) = Q(2-r)$. Since this holds for infinitely many r, it must be a polynomial identity. So $Q(1+x)$ is an even polynomial, and so we can let $Q(x) = A((x-1)^2)$ for some polynomial A.

Retruning to the origional functional equation, we note that:

$$P(\sqrt{2}x) = Q(2x^2) = A(4x^4 - 4x^2 + 1)$$

And:

$$P(x + \sqrt{1-x^2}) = Q(1 + 2x\sqrt{1-x^2}) = A(4x^2 - 4x)$$

So for all $x \in (0,1)$, $A(4x^2 - 4x^4) = A(1 - (4x^2 - 4x^4))$. Since there are infinitely many possible values of $4x^2 - 4x^4$, we must have that $A(x) = A(1-x)$ is a polynomial identity. So $A(\frac{1}{2} + x)$ is an even polynomial, and so $A(x) = B((x - \frac{1}{2})^2)$ for some polynomial B.

Thus, we have that $P(x)$ must satisfy $P(x) = B(P_0(x))$ for some polynomial B, where $P_0(x) = [(x^2-1)^2 - \frac{1}{2}]^2$. It is easy to verify that P_0 is indeed a solution, so all polynomials in P_0 must be solutions as well. The proof is complete.

4. Let $P(x)$ and $Q(x)$ be arbitrary polynomials with real coefficients, and let d be the degree of $P(x)$. Assume that $P(x)$ is not the zero polynomial. Prove that there exist polynomials $A(x)$ and $B(x)$ with real coefficients, such that:

   (i) both A and B have degree at most $d/2$, and

   (ii) at most one of A and B is the zero polynomial, and

   (iii) $\frac{A(x)+Q(x)B(x)}{P(x)}$ is a polynomial with real coefficients. That is, there is some polynomial $C(x)$ with real coefficients such that $A(x) + Q(x)B(x) = P(x)C(x)$.

Solution 1. We will prove the following stronger statement: given any nonzero poylnomials P and Q with $\deg P \le \deg Q$ and a nonnegative integer ℓ with $\ell < \min(\deg P, \deg Q)$ we can find polynomials A, B, C not all zero such that

$$A(x) + B(x)Q(x) = C(x)P(x)$$

and

$$\begin{aligned}\deg A &\le \ell \\ \deg B &\le \deg P - \ell - 1 \\ \deg C &\le \deg Q - \ell - 1.\end{aligned}$$

The proof is by strong induction on $\deg P + \deg Q$. By the division algorithm, we may write

$$Q(x) = D(x)P(x) + R(x)$$

with $\deg R < \deg P$ and $\deg D = \deg Q - \deg P$ with D nonzero. If $\deg R \le \ell$, we may take $A = -R, \quad B = 1 \quad C = D$. It's easy to check that B is nonzero and

$$\begin{aligned}\deg A &= \deg R \le \ell \\ \deg B &= 0 \le \deg P - \ell - 1 \\ \deg C &= \deg D = \deg Q - \deg P \le \deg Q - \ell - 1\end{aligned}$$

so we're done.

Now suppose $\deg R > \ell$. By the induction hypothesis on R and P, there exist A', B', C' with

$$\begin{aligned}\deg A' &\le \ell \\ \deg B' &\le \deg R - \ell - 1 \\ \deg C' &\le \deg P - \ell - 1.\end{aligned}$$

Since $R(x) = Q(x) - D(x)P(x)$, we have

$$\begin{aligned}A'(x) + B'(x)P(x) &= C'(x)R(x) \\ A'(x) + (B'(x) + C'(x)D(x))P(x) &= C'(x)Q(x).\end{aligned}$$

We can check that

$$\begin{aligned}\deg(B' + C'D) &= \max(\deg B', \deg C' + \deg D) \\ &\le \max(\deg Q - \ell - 1, (\deg P - \ell - 1) + (\deg Q - \deg P)) \\ &= \deg Q - \ell - 1.\end{aligned}$$

So by setting $A = -A', B = C', C = B' + C'D$, we've found valid A, B, C for $\deg R > \ell$, completing the induction.

The problem statement follows now follows by setting $\ell = \left\lfloor \frac{d}{2} \right\rfloor$.

Solution 2. Given polynomials P, Q with $Q \neq 0$, we let $[P]_Q$ denote the remainder of P upon division by Q. By comparing degrees and working modulo Q, we find that $[uP_1 + vP_2]_Q = u[P_1]_Q + v[P_2]_Q$ for any $P_1, P_2 \in F[x]$ and constants $u, v \in F$.

Now set $d' = \lfloor d/2 \rfloor$ for convenience, so we need $B = b_0 + b_1 x + \cdots + b_{d'} x^{d'}$, with $b_i \in F$ not all zero, such that $[CB]_M$ has $[x^{d'+1}], \ldots, [x^{d-1}]$ coefficients all zero, but $[x^0], \ldots, [x^{d'}]$ coefficients *not* all zero. However, $\gcd(C, M) = 1$ means $CP \not\equiv 0 \pmod{M}$ for any $P \not\equiv 0 \pmod{M}$, so as long as we satisfy the former condition, we satisfy the latter as well.

Since

$$[C(b_0 + \cdots + b_{d'} x^{d'})]_M = b_0[C]_M + b_1[Cx]_M + \cdots + b_{d'}[Cx^{d'}]_M,$$

we simply need a nontrivial solution to the homogeneous linear system

$$\begin{bmatrix} [x^{d'+1}][C]_M & [x^{d'+1}][Cx]_M & \cdots & [x^{d'+1}][Cx^{d'}]_M \\ \vdots & \vdots & \ddots & \vdots \\ [x^{d-1}][C]_M & [x^{d-1}][Cx]_M & \cdots & [x^{d-1}][Cx^{d'}]_M \end{bmatrix} \begin{bmatrix} b_0 \\ \vdots \\ b_{d'} \end{bmatrix} = \mathbf{0}_{\mathbf{d-1-d',1}}$$

of $k = d - 1 - d'$ equations in $n = d' + 1$ variables. But $d' = \lfloor \frac{d}{2} \rfloor \geq \frac{d}{2} - 1$ yields

$$n - k = 2(d' + 1) - d \geq 2\frac{d+1}{2} - d > 0,$$

so we in fact have a *underdetermined homogeneous linear system*. As these systems always have nontrivial solutions, we're done. (Proof sketch: If $k = 1$, the result is clear. Otherwise, eliminate 1 variable by substituting from one of its equations into each of the others, and note that we cannot get any "conflicts"; the system now reduces to the $n - 1$ variable, $\leq k - 1$ equation case.)

5. Find the maximum number E such that the following holds: there is an edge-colored graph with 60 vertices and E edges, with each edge colored either red or blue, such that in that coloring, there are no monochromatic cycles of length 3 and no monochromatic cycles of length 5.

Solution 1. The answer is 1350. We begin with the upper bound.

Lemma 1. A graph G which has all edges colored with red or blue and has no monochromatic 3 or 5 cycle has no K_5 as a subgraph.

Proof. Suppose that we have a K_5.. Let 1,2,3,4 and 5 denote the vertices of the K_5. First, consider the case in which some vertex of the K_5 has at least three incident edges of the same color; without loss of generality, color the edges (12), (13), (14) red. Since the cycles (123), (124), (134) are not monochromatic, this means that the edges (23), (24) and (34) have to be blue and therefore the graph has the blue 3-cycle (234), a contradiction.

Hence, for all $i \in \{1, 2, 3, 4, 5\}$, vertex i has exactly 2 incident red edges and 2 incident blue edges in the K_5. Consequently, every vertex must be in exactly one red cycle. However, because 3- and 5-cycles are not allowed, we immediately see that this is impossible, again a contradiction. The lemma follows. □

Now, as G has no K_5, by Turan's Theorem we have

$$|E(G)| \leq \frac{3}{4} \cdot \frac{60^2}{2} = 1350,$$

establishing the upper bound.

It is left to exhibit a red-blue edge-colored graph G with 60 vertices, 1350 edges, and no monochromatic 3- or 5-cycles. Consider four disjoint sets S_1, S_2, S_3, S_4, of 15 vertices each. Draw a blue edge from each vertex of S_1 to each other vertex in S_2 and from each vertex of S_3 to each other vertex of S_4. Also, draw a red edge from each vertex in S_1 or S_2 to each other vertex in S_3 or S_4. Taking G to be the union of all of these edges and vertices, we are done.

6. Suppose we have distinct positive integers a, b, c, d, and an odd prime p not dividing any of them, and an integer M such that if one considers the infinite sequence

$$\begin{aligned} &ca - db \\ &ca^2 - db^2 \\ &ca^3 - db^3 \\ &ca^4 - db^4 \\ &\cdots \end{aligned}$$

and looks at the highest power of p that divides each of them, these powers are not all zero, and are all at most M. Prove that there exists some T (which may depend on a, b, c, d, p, M) such that whenever p divides an element of this sequence, the maximum power of p that divides that element is exactly p^T.

Solution 1. Take $K \geq 1$ with $v_p(ca^K - db^K) = M$ maximal. Assume for the sake of contradiction that $N = v_p(a^{p-1} - b^{p-1}) \leq M$, so $M \geq N \geq 1$ by Fermat's Little Theorem. Since p is an odd prime not dividing ab, the lifting the exponent lemma yields

$$v_p(a^L - b^L) = v_p(a^{p-1} - b^{p-1}) + v_p(p^{M-N}) = N + (M - N) = M,$$

where $L = p^{M-N}(p-1)$. Now write $ca^K = db^K + p^M r$ and $a^L = b^L + p^M s$ for integers r, s not divisible by p, and observe that for any positive integer u,

$$\begin{aligned} c\left(\frac{a}{b}\right)^{K+uL} &= \left(d + p^M \frac{r}{b^K}\right)\left(1 + p^M \frac{s}{b^L}\right)^u \\ &\equiv \left(d + p^M \frac{r}{b^K}\right)\left(1 + up^M \frac{s}{b^L}\right) \\ &\equiv d + p^M(rb^{-K} + udsb^{-L}) \pmod{p^{2M}}. \end{aligned}$$

But $p \nmid dsb$, so setting $u \equiv -rb^{-K}(dsb^{-L})^{-1} \pmod{p}$, we obtain $p^{M+1} \mid (ca^{K+uL} - db^{K+uL})$ (note that $2M \geq M + 1$), contradicting the choice of K.

Thus $N > M$. If $v_p(ca^k - db^k) > 0$, then

$$\frac{(ca^k)^{p-1} - (db^k)^{p-1}}{ca^k - db^k} \equiv (p-1)(db^k)^{p-1} \not\equiv 0 \pmod{p},$$

so $v_p(c^{p-1}a^{k(p-1)} - d^{p-1}b^{k(p-1)}) = v_p(ca^k - db^k) \leq M < N$. But

$$p^N \mid (a^{p-1} - b^{p-1}) \mid (a^{k(p-1)} - b^{k(p-1)}),$$

so $c^{p-1}a^{k(p-1)} - d^{p-1}b^{k(p-1)} \equiv b^{k(p-1)}(c^{p-1} - d^{p-1}) \pmod{p^N}$. Since $p \nmid b$, it follows that $v_p(c^{p-1} - d^{p-1}) = v_p(ca^k - db^k)$. Thus $t = v_p(c^{p-1} - d^{p-1})$ suffices.

Alternatively, if k_0 is the smallest k with $p \mid ca^k - db^k$, then $p \mid ca^k - db^k$ iff $k \equiv k_0 \pmod{\ell}$, where $\ell = \mathrm{ord}_p(ab^{-1}) \mid p - 1$. Since $v_p(a^\ell - b^\ell) = v_p(a^{p-1} - b^{p-1}) = N > M$, $v_p(ca^{k_0+i\ell} - db^{k_0+i\ell}) = v_p(b^{i\ell}(ca^{k_0} - db^{k_0})) = v_p(ca^{k_0} - db^{k_0})$.