2016–2017 USA Team Selection Test #1

Thursday, December 8, 2016

Time limit: 4.5 hours. Each question is worth 7 points.

1. In a sports league, each team uses a set of at most t signature colors. A set S of teams is color-identifiable if one can assign each team in S one of their signature colors, such that no team in S is assigned any signature color of a different team in S.

   For all positive integers n and t, determine the maximum integer $g(n,t)$ such that in a sports league where n is exactly the size of the union of all signature color sets, one can always find a color-identifiable set of at least $g(n,t)$ teams.

2. Let ABC be an acute scalene triangle with circumcenter O, and let T be on line BC such that $\angle TAO = 90^\circ$. The circle with diameter $\overline{AT}$ intersects the circumcircle of $\triangle BOC$ at two points A_1 and A_2, where $OA_1 < OA_2$. Points B_1, B_2, C_1, C_2 are defined analogously.

   (a) Prove that $\overline{AA_1}$, $\overline{BB_1}$, $\overline{CC_1}$ are concurrent.

   (b) Prove that $\overline{AA_2}$, $\overline{BB_2}$, $\overline{CC_2}$ are concurrent on the Euler line of triangle ABC.

3. Let $P, Q \in \mathbb{R}[x]$ be relatively prime nonconstant polynomials. Show that there can be at most three real numbers λ such that $P + \lambda Q$ is the square of a polynomial.

2016–2017 USA Team Selection Test #2

Thursday, January 19, 2017

Time limit: 4.5 hours. Each question is worth 7 points.

1. You are cheating at a trivia contest. For each question, you can peek at each of the $n > 1$ other contestants' guesses before writing down your own. For each question, after all guesses are submitted, the emcee announces the correct answer. A correct guess is worth 0 points. An incorrect guess is worth -2 points for other contestants, but only -1 point for you, because you hacked the scoring system. After announcing the correct answer, the emcee proceeds to read out the next question. Show that if you are leading by 2^{n-1} points at any time, then you can surely win first place.

2. Let ABC be a triangle with altitude $\overline{AE}$. The A-excircle touches $\overline{BC}$ at D, and intersects the circumcircle at two points F and G. Prove that one can select points V and N on lines DG and DF such that quadrilateral $EVAN$ is a rhombus.

3. Prove that there are infinitely many triples (a, b, p) of positive integers with p prime, $a < p$, and $b < p$, such that $(a + b)^p - a^p - b^p$ is a multiple of p^3.