2015–2016 USA Team Selection Test #1

Thursday, December 10, 2015

Time limit: 4 hours. Each question is worth 7 points.

1. Let n be a positive integer, and let $f_1, \ldots, f_k$ be bijections (one-to-one and onto maps) from the set $\{1, 2, \ldots, n\}$ to itself. For any f_i and any $x \in \{1, 2, \ldots, n\}$, define the f_i-*orbit* of x to be the set of elements
$$\{x, f_i(x), f_i(f_i(x)), \ldots\}.$$
Define $c(f_i)$ to be the number of distinct f_i orbits. For instance, the bijection f on $\{1, 2, 3\}$ defined by $f(1) = 2$, $f(2) = 1$, and $f(3) = 3$ has two orbits, $\{1, 2\}$ and $\{3\}$, and so $c(f) = 2$.
Let $f_1 \circ f_2 \circ \cdots \circ f_k$ denote the composition of these functions, defined by $f_1 \circ f_2 \circ \cdots \circ f_k(x) = f_1(f_2(\cdots f_k(x) \cdots))$. Prove that:
$$\left(\sum_{i=1}^{k} c(f_i)\right) - c(f_1 \circ f_2 \circ \cdots \circ f_k) \le n(k-1).$$

2. Let ABC be a non-isosceles triangle with circumcircle Ω, and suppose the incircle of ABC touches BC at D. The angle bisector of $\angle A$ meets BC and Ω at K and M. The circumcircle of $\triangle DKM$ intersects the A-excircle (excircle of $\triangle ABC$ opposite A) at S_1 and S_2, and it intersects circle Ω at M and T. Prove that line AT passes through either S_1 or S_2.

3. Define $\theta_p : \mathbb{F}_p[x] \to \mathbb{F}_p[x]$ by:
$$\theta_p\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} a_i x^{p^i}.$$
Prove that if F and G are non-zero polynomials in $\mathbb{F}_p[x]$, then:
$$\gcd(\theta_p(F), \theta_p(G)) = \theta_p(\gcd(F, G)).$$

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments: straightedge, compass, set square (a.k.a., right triangle). Failure to meet any of these requirements will result in a 1-point automatic deduction.

2015–2016 USA Team Selection Test #2

Thursday, January 21, 2016

Time limit: 4 hours. Each question is worth 7 points.

1. Let $\sqrt{3} = 1.b_1b_2b_3\dots_{(2)}$ be the binary representation of $\sqrt{3}$. Prove that for any positive integer n, at least one of the digits $b_n, b_{n+1}, \dots, b_{2n}$ equals 1.

2. Let $n \geq 4$ be a natural number, and let $[n]$ denote the set $\{1, \dots, n\}$. Find all functions $W : [n]^2 \to \mathbb{R}$ such that for every partition $[n] = A \cup B \cup C$ into disjoint sets,

$$\sum_{a \in A} \sum_{b \in B} \sum_{c \in C} W(a,b)W(b,c) = |A||B||C|.$$

3. Let ABC be an acute scalene triangle and let P be a point in its interior. Let A_1, B_1, C_1 be projections of P onto triangle sides BC, CA, AB, respectively. Find the locus of points P such that AA_1, BB_1, CC_1 are concurrent and $\angle PAB + \angle PBC + \angle PCA = 90°$.