2014–2015 USA Team Selection Test #1

Thursday, December 11, 2014

Time limit: 4 hours. Each question is worth 7 points.

1. Let ABC be a non-isosceles triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Denote by M the midpoint of $\overline{BC}$. Let Q be a point on the incircle such that $\angle AQD = 90^\circ$. Let P be the point inside the triangle on line AI for which $MD = MP$. Prove that either $\angle PQE = 90^\circ$ or $\angle PQF = 90^\circ$.

2. Prove that for every $n \in \mathbb{N}$, there exists a set S of n positive integers such that for any two distinct $a, b \in S$, $a - b$ divides a and b but none of the other elements of S.

3. A physicist encounters 2015 atoms called usamons. Each usamon either has one electron or zero electrons, and the physicist can't tell the difference. The physicist's only tool is a diode. The physicist may connect the diode from any usamon A to any other usamon B. (This connection is directed.) When she does so, if usamon A has an electron and usamon B does not, then the electron jumps from A to B. In any other case, nothing happens. In addition, the physicist cannot tell whether an electron jumps during any given step. The physicist's goal is to isolate two usamons that she is 100% sure are currently in the same state. Is there any series of diode usage that makes this possible?

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments: straightedge, compass, set square (a.k.a., right triangle). Failure to meet any of these requirements will result in a 1-point automatic deduction.

2014–2015 USA Team Selection Test #2

Thursday, January 22, 2015

Time limit: 4 hours. Each question is worth 7 points.

1. Prove or disprove: if $f : \mathbb{Q} \to \mathbb{Q}$ satisfies

$$f(x+y) - f(x) - f(y) \in \mathbb{Z}$$

   for all rationals x and y, then there exists $c \in \mathbb{Q}$ such that $f(x) - cx \in \mathbb{Z}$ for all $x \in \mathbb{Q}$. Here, $\mathbb{Z}$ denotes the set of integers, and $\mathbb{Q}$ denotes the set of rationals.

2. A tournament is a directed graph for which every (unordered) pair of distinct vertices has a single directed edge from one vertex to the other. Let us define a proper directed-edge-coloring to be an assignment of a color to every (directed) edge, so that for every pair of directed edges $\overrightarrow{uv}$ and $\overrightarrow{vw}$ (with the first edge oriented into, and the second edge oriented out of a common endpoint v), those two edges are in different colors.

   Note that it is permissible for edge pairs of the form $\overrightarrow{vu}$ and $\overrightarrow{vw}$ to be the same color, as well as for edge pairs of the form $\overrightarrow{uv}$ and $\overrightarrow{wv}$ to be the same color.

   The directed-edge-chromatic-number of a tournament is defined to be the minimum total number of colors that can be used in order to create a proper directed-edge-coloring. For each n, determine the minimum directed-edge-chromatic-number over all tournaments on n vertices.

3. Let ABC be a non-equilateral triangle and let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively. Let S be a point lying on the Euler line of ABC. Denote by X, Y, Z the second intersections of each of M_aS, M_bS, M_cS with the nine-point circle, respectively. Prove that AX, BY, CZ are concurrent.

   Recall that the Euler line is the line through the circumcenter and orthocenter, and the nine-point circle is the circumcircle of M_a, M_b, and M_c.

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