USA Team Selection Test for IMO 2015

Problems and Solutions

56th IMO 2015 at Chiang Mai, Thailand

1 Problems

Thursday, December 11, 2014

1. Let ABC be a non-isosceles triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Denote by M the midpoint of $\overline{BC}$. Let Q be a point on the incircle such that $\angle AQD = 90^\circ$. Let P be a point inside the triangle on line AI for which $MD = MP$. Prove that either $\angle PQE = 90^\circ$ or $\angle PQF = 90^\circ$.

   Evan Chen

2. Prove that for every $n \in \mathbb{N}$, there exists a set S of n positive integers such that for any two distinct $a, b \in S$, $a - b$ divides a and b but none of the other elements of S.

   Iurie Boreico

3. A physicist encounters 2015 atoms called usamons. Each usamon either has one electron or zero electrons, and the physicist can't tell the difference. The physicist's only tool is a diode. The physicist may connect the diode from any usamon A to any other usamon B. (This connection is directed.) When she does so, if usamon A has an electron and usamon B does not, then the electron jumps from A to B. In any other case, nothing happens. In addition, the physicist cannot tell whether an electron jumps during any given step. The physicist's goal is to isolate two usamons that she is 100% sure are currently in the same state. Is there any series of diode usage that makes this possible?

   Linus Hamilton

Thursday, January 15, 2015

1. Prove or disprove: if $f : \mathbb{Q} \to \mathbb{Q}$ satisfies

$$f(x+y) - f(x) - f(y) \in \mathbb{Z}$$

for all rationals x and y, then there exists $c \in \mathbb{Q}$ such that $f(x) - cx \in \mathbb{Z}$ for all $x \in \mathbb{Q}$. Here, $\mathbb{Z}$ denotes the set of integers, and $\mathbb{Q}$ denotes the set of rationals.

   Victor Wang

2. A tournament is a directed graph for which every (unordered) pair of distinct vertices has a single directed edge from one vertex to the other. Let us define a proper directed-edge-coloring to be an assignment of a color to every (directed) edge, so that for every pair of directed edges $\overrightarrow{uv}$ and $\overrightarrow{vw}$ (with the first edge oriented into, and the second edge oriented out of a common endpoint v), those two edges are in different colors.

   Note that it is permissible for edge pairs of the form $\overrightarrow{vu}$ and $\overrightarrow{vw}$ to be the same color, as well as for edge pairs of the form $\overrightarrow{uv}$ and $\overrightarrow{wv}$ to be the same color.

   The directed-edge-chromatic-number of a tournament is defined to be the minimum total number of colors that can be used in order to create a proper directed-edge-coloring. For each n, determine the minimum directed-edge-chromatic-number over all tournaments on n vertices.

   Po-Shen Loh

3. Let ABC be a non-equilateral triangle and let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively. Let S be a point lying on the Euler line of ABC. Denote by X, Y, Z the second intersections of each of M_aS, M_bS, M_cS with the nine-point circle, respectively. Prove that AX, BY, CZ are concurrent.

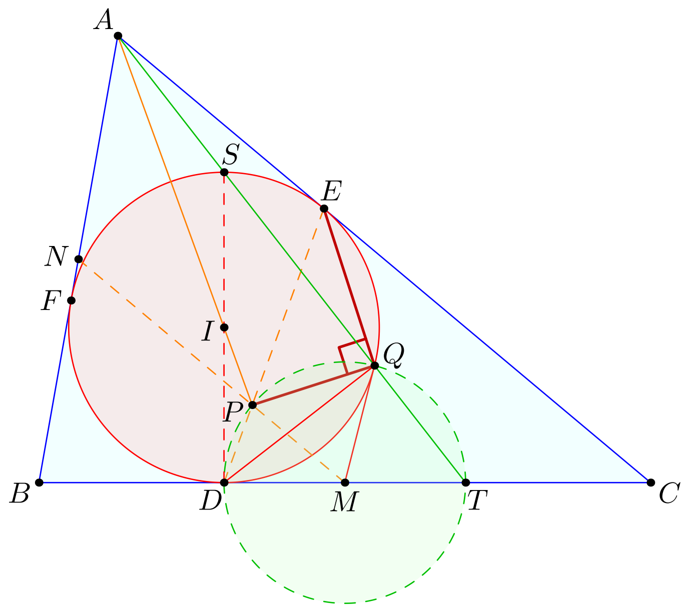
   Recall that the Euler line is the line through the circumcenter and orthocenter, and the nine-point circle is the circumcircle of M_a, M_b, and M_c.

   Ivan Borsenco

2 Solutions for the December TST

Problem 1

Assume without loss of generality that $AB < AC$; we show $\angle PQE = 90^\circ$.



First, we claim that D, P, E are collinear. Let N be the midpoint of $\overline{AB}$. It is well-known that the three lines MN, DE, AI are concurrent at a point (see for example problem 6 of USAJMO 2014). Let P' be this intersection point, noting that P' actually lies on segment DE. Then P' lies inside $\triangle ABC$ and moreover

$$\triangle DP'M \sim \triangle DEC$$

so $MP' = MD$. Hence $P' = P$, proving the claim.

Let S be the point diametrically opposite D on the incircle, which is also the second intersection of $\overline{AQ}$ with the incircle. Let $T = \overline{AQ} \cap \overline{BC}$. Then T is the contact point of the A-excircle; consequently,

$$MD = MP = MT$$

and we obtain a circle with diameter $\overline{DT}$. Since $\angle DQT = \angle DQS = 90^\circ$ we have Q on this circle as well.

As $\overline{SD}$ is tangent to the circle with diameter $\overline{DT}$, we obtain

$$\angle PQD = \angle SDP = \angle SDE = \angle SQE.$$

Since $\angle DQS = 90^\circ$, $\angle PQE = 90^\circ$ too.

Problem 2

The idea is to look for a sequence $d_1, \dots, d_{n-1}$ of "differences" such that the following two conditions hold. Let $s_i = d_1 + \cdots + d_{i-1}$, and $t_{i,j} = d_i + \cdots + d_j$ for $i \le j$.

(i) No two of the $t_{i,j}$ divide each other.

(ii) There exists an integer a satisfying the CRT equivalences

$$a \equiv -s_i \pmod{t_{i,j}} \quad \forall i \le j$$

Then the sequence $a+s_1$, $a+s_2$, …, $a+s_n$ will work. For example, when $n = 3$ we can take $(d_1, d_2) = (2, 3)$ giving

$$\overbrace{10 \underbrace{\qquad}_{2} 12 \underbrace{\qquad}_{3} 15}^{5}$$

because the only conditions we need satisfy are

$$\begin{aligned} a &\equiv 0 \pmod{2} \\ a &\equiv 0 \pmod{5} \\ a &\equiv -2 \pmod{3}. \end{aligned}$$

But with this setup we can just construct the d_i inductively. To go from n to $n+1$, take a $d_1, \dots, d_{n-1}$. Let M be a multiple of $\prod_{i=1}^{n-1} d_i$, and p a prime not dividing M. Then we claim that $d_1 M, d_2 M, \dots, d_{n-1} M, p$ is such a difference sequence. For example, the previous example extends as follows.

$$\overbrace{a \underbrace{\qquad}_{600} b \overbrace{\underbrace{\qquad}_{900} c \underbrace{\qquad}_{7}}^{907} d}^{1507}$$

The new numbers p, $p + Md_{n-1}$, $p + Md_{n-2}$, … are all relatively prime to everything else. Hence (i) still holds. To see that (ii) still holds, just note that we can still get a family of solutions for the first n terms, and then the last $n+1$th term can be made to work by Chinese Remainder Theorem since all the new $p + Md_k$ are coprime to everything.

Problem 3

The answer is no. Call the usamons $U_1, \dots, U_m$ (here $m = 2015$). Consider models M_k of the following form: $U_1, \dots, U_k$ are all charged for some $0 \le k \le m$ and the other usamons are not charged. Note that for any pair there's a model where they are different states, by construction.

We can consider the physicist as acting on these $m+1$ models simultaneously, and trying to reach a state where there's a pair in all models which are all the same charge. (This is a necessary condition for a winning strategy to exist.)

But we claim that any diode operation $U_i \to U_j$ results in the $m+1$ models being an isomorphic copy of the previous set. If $i < j$ then the diode operation can be interpreted as just swapping U_i with U_j, which doesn't change anything. Moreover if $i > j$ the operation never does anything. The conclusion follows from this.

3 Solutions for the January TST

Problem 1

The statement is false. To show this, we will construct explicitly a function $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x+y) - f(x) - f(y) \in \mathbb{Z} \tag{1}$$

for all $x, y \in \mathbb{Z}$ but such that there is no $c \in \mathbb{Q}$ with $f(x) - cx \in \mathbb{Z}$ for all $x \in \mathbb{Q}$.

For a rational number x, denote by $\{x\}$ the fractional part of x (that is, the number $x - \lfloor x \rfloor$). Let $Q \subset \mathbb{Q}$ be the set of rational numbers whose denominator is a power of 2, and let $R \subset \mathbb{Q}$ be the set of rational numbers whose denominator is odd. The sets Q and R are both closed under addition and additive inverse. We have the following lemma:

Lemma 1. *Every rational number x can be expressed in the form $q + r$, where $q \in Q$ and $r \in R$. Furthermore, if $x = q + r$ and $x = q' + r'$ are two such expressions (where $q, q' \in Q$ and $r, r' \in R$), then $q - q'$ is an integer.*

Proof. First, we show that x can be expressed in the desired form. Write $x = m/n$, where m and n are relatively prime integers. Then, write n as a product $n_1 n_2$, where n_1 is a power of 2 and n_2 is odd. Since n_1 and n_2 are relatively prime, there are integers a, b such that $an_1 + bn_2 = 1$. Take

$$q = \frac{mb}{n_1}$$

and

$$r = \frac{ma}{n_2}.$$

By construction, we have $q \in Q$ and $r \in R$ so it remains to check that $q + r = x$. We compute

$$\begin{aligned} q + r &= \frac{mb}{n_1} + \frac{ma}{n_2} \\ &= \frac{m(an_1 + bn_2)}{n_1 n_2} \\ &= \frac{m}{n} \\ &= x \end{aligned}$$

as desired.

Now, suppose that $x = q + r = q' + r'$ where $q, q' \in Q$, $r, r' \in R$; we wish to show that $q - q'$ is an integer. Since Q contains both q and q' and is closed under addition and additive inverse, we conclude that $q - q' \in Q$. Similarly, we have $r' - r \in R$. But $q - q' = r' - r$, so it is in $Q \cap R$. That is, the denominator of $q - q'$ is both odd and a power of 2. The only odd power of 2 is 1, so $q - q'$ has a denominator of 1; in other words, it is an integer, as desired. $\square$

Define the function f as follows: for $x \in \mathbb{Q}$, let $f(x) = \{q\}$, where $x = q + r$ is a decomposition with $q \in Q$ and $r \in R$. By Lemma 1, this uniquely defines the function f.

We check property (1). Let $x, y \in Q$; we wish to show that

$$f(x+y) - f(x) - f(y) \in \mathbb{Z}.$$

By Lemma 1, we can write $x = q+r$ and $y = q'+r'$ where $q, q' \in Q$ and $r, r' \in R$. Adding these two equations, we obtain a decomposition

$$x + y = (q + q') + (r + r')$$

and $q + q' \in Q$ and $r + r' \in R$. By the definition of the function f, we have

$$\begin{aligned} f(x) &= \{q\} \\ f(y) &= \{q'\} \\ f(x+y) &= \{q + q'\} \end{aligned}$$

so

$$f(x+y) - f(x) - f(y) = \{q + q'\} - \{q\} - \{q'\}$$

is an integer, as desired.

Now, assume for the sake of contradiction that there is $c \in \mathbb{Q}$ such that $f(x) - cx \in \mathbb{Z}$ for all $x \in \mathbb{Q}$. Fix an odd integer $\ell > |c|$. We have a decomposition

$$\frac{1}{\ell} = 0 + \frac{1}{\ell}$$

and $0 \in Q$ and $1/\ell \in R$, so

$$f\left(\frac{1}{\ell}\right) = 0.$$

Since

$$f\left(\frac{1}{\ell}\right) - \frac{c}{\ell} = -\frac{c}{\ell}$$

is an integer, we must have $c = 0$. Therefore $f(x) \in \mathbb{Z}$ for all $x \in \mathbb{Q}$. On the other hand, the decomposition

$$\frac{1}{2} = \frac{1}{2} + 0$$

with $1/2 \in Q$ and $0 \in R$ yields

$$f\left(\frac{1}{2}\right) = \frac{1}{2},$$

so this is a contradiction.

Problem 2

Let $f(n)$ be the answer to the problem. We claim that $f(n) = \lceil \log n \rceil$ for all $n \geq 1$. The proof is divided into showing that this quantity is both an upper and a lower bound.

Lower Bound. We show that every tournament on n vertices has directed-edge-chromatic number at least $\lceil \log n \rceil$. We prove this by strong induction on n. As our base case, if $n = 1$, then clearly $f(1) \geq 0 = \lceil \log 1 \rceil$. Now, for $n \geq 2$, assume by the strong inductive hypothesis that $f(k) \geq \lceil \log k \rceil$ for all $k < n$. Consider any proper directed-edge-coloring of an n-vertex directed graph (vertex set V) with $f(n) \geq 1$ colors. Pick an arbitrary color, and let U_1 be the set of vertices u for which their exists an edge $\overrightarrow{uv}$ of that color, and let U_2 be the set of vertices u for which their exists an edge of the form $\overrightarrow{vu}$. Clearly since the coloring is proper $U_1 \cap U_2 = \emptyset$. Thus, there exists $i \in \{1, 2\}$ such that U_i has size at most $n/2$. Notice that $V \setminus U_i$ has size at least $n/2$ and also has no edges of the originally chosen color. Thus, the graph induced by $V \setminus U_i$ has a proper directed-edge-coloring of $f(n) - 1$ colors. By the inductive hypothesis, this implies that

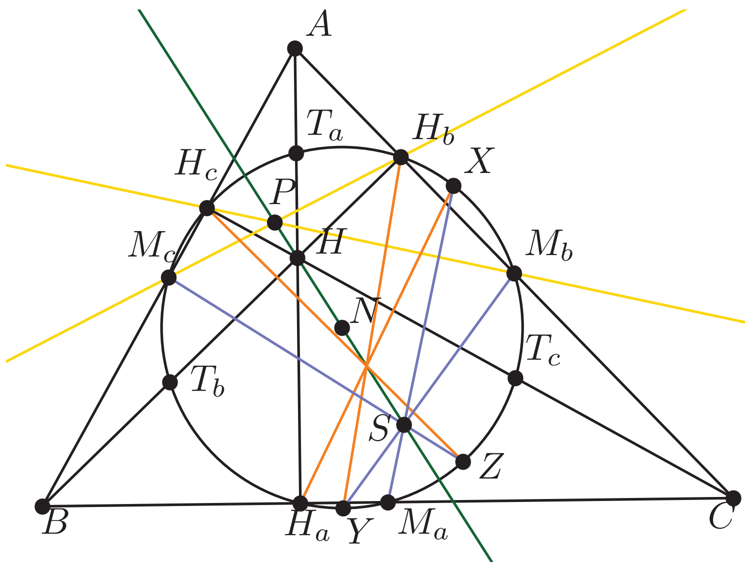
$$f(n) - 1 \geq f(|V \setminus U_i|) \geq \lceil \log |V \setminus U_i| \rceil \geq \lceil \log(n/2) \rceil = \lceil \log n \rceil - 1.$$

Therefore, $f(n) \geq \lceil \log n \rceil$.

Upper Bound. We exhibit a proper directed-edge-coloring with $\lceil \log n \rceil$ colors of a tournament with n vertices. Consider the tournament with vertices $0, \ldots, n-1$ such that there is a directed edge from i to j if and only if $i > j$. Label our colors $0, \ldots, \lceil \log n \rceil - 1$. Color the edge $\overrightarrow{ij}$ the largest k such that i has a 1 and j has a 0 in the 2^kth place of their binary representations. Such a k always exists and is at most $\lceil \log n \rceil - 1$ because $i > j$ and $n - 1$ requires at most $\lceil \log n \rceil$ digits in its binary representation. Assume for sake of contradiction that this is not a proper directed-edge-coloring. Then, there would exist edge $\overrightarrow{ij}$ and $\overrightarrow{jk}$ of color ℓ. Thus, j would have both 0 and 1 in the 2^ℓth place of its binary representation, contradiction. Hence, this exhibited coloring is proper so $f(n) \le \lceil \log n \rceil$.

Thus, the minimal directed-edge-chromatic number among tournaments on n vertices is $\lceil \log n \rceil$.

Problem 3



Let H, N be the orthocenter and nine-point center, respectively, of ABC. Let H_a, H_b, H_c be the feet of the altitudes from A, B, C, respectively, and let T_A, T_B, T_C be the midpoints of AH, BH, CH respectively.

Let $P = M_cH_b \cap M_bH_c$. Pascal's theorem on hexagon $M_bT_bH_bM_cT_cH_c$ gives that P is collinear with $M_bT_b \cap M_cT_c = N$ and $T_bH_b \cap T_cH_c = H$. So P is on the Euler line NH. Then Pascal's theorem on $M_bYH_bM_cZH_c$ gives that $H_bY \cap H_cZ$ is collinear with $M_bY \cap M_cZ = S$ and $M_bH_c \cap H_bM_c = P$, so H_bY and H_cZ meet on the Euler line. By symmetry H_bY also meets H_aX on the Euler line, so H_aX, H_bY, H_cZ concur.

Using trig Ceva, the concurrence of H_aX, H_bY, H_cZ gives

$$\prod \frac{\sin \angle H_cH_aX}{\sin \angle H_bH_aX} = 1,$$

and the concurrence of XM_a, YM_b, ZM_c gives

$$\prod \frac{\sin \angle M_c M_a X}{\sin \angle M_b M_a X} = 1.$$

Thus

$$\begin{aligned}
\prod \frac{\sin \angle BAX}{\sin \angle CAX} &= \prod \frac{\sin \angle M_c AX}{\sin \angle M_b AX} = \prod \frac{\frac{M_c X}{AX} \sin \angle AM_c X}{\frac{M_b X}{AX} \sin \angle AM_b X} \\
&= \prod \frac{\sin \angle AM_c X}{\sin \angle AM_b X} \prod \frac{M_c X}{M_b X} = \prod \frac{\sin \angle H_c H_a X}{\sin \angle H_b H_a X} \\
&= \prod \frac{\sin \angle M_c M_b X}{\sin \angle M_b M_c X} = \prod \frac{\sin \angle M_c M_b X}{\sin \angle M_b M_c X} \\
&= \prod \frac{\sin \angle M_c M_a X}{\sin \angle M_b M_a X} = 1.
\end{aligned}$$

Thus by trig Ceva again, we have that AX, BY, CZ are concurrent as wanted.