2013–2014 USA Team Selection Test #1

Thursday, December 12, 2013

Congratulations on making it this far! Your prize is the following **3-question, 4-hour** problem set. That was meant in full seriousness: we hope you will find these problems as intriguing as we did. Each is worth the usual 7 points. Good luck, and have fun.

1. Let ABC be an acute triangle, and let X be a variable interior point on the minor arc $\widehat{BC}$. Let P and Q be the feet of the perpendiculars from X to lines CA and CB, respectively. Let R be the intersection of line PQ and the perpendicular from B to AC. Let ℓ be the line through P parallel to XR. Prove that as X varies along minor arc $\widehat{BC}$, the line ℓ always passes through a fixed point. (Specifically: prove that there is a point F, determined by triangle ABC, such that no matter where X is on arc $\widehat{BC}$, line ℓ passes through F.)

2. Let $a_1, a_2, a_3, \ldots,$ be a sequence of integers, with the property that every consecutive group of a_i's averages to a perfect square. More precisely, for every positive integers n and k, the quantity
$$\frac{a_n + a_{n+1} + \ldots + a_{n+k-1}}{k}$$
is always the square of an integer. Prove that the sequence must be constant (all a_i are equal to a fixed perfect square).

3. Let n be an even positive integer, and let G be an n-vertex graph with exactly $\frac{n^2}{4}$ edges, where there are no loops or multiple edges (each unordered pair of distinct vertices is joined by either 0 or 1 edge). An unordered pair of distinct vertices $\{x, y\}$ is said to be *amicable* if they have a common neighbor (there is a vertex z such that xz and yz are both edges). Prove that G has at least $2\binom{n/2}{2}$ unordered pairs of vertices which are amicable.

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments: straightedge, compass, set square (a.k.a., right triangle). Failure to meet any of these requirements will result in a 1-point automatic deduction.

2013–2014 USA Team Selection Test #2

Thursday, January 23, 2014

Welcome back! This is a 4-hour problem set. Each question is worth 7 points. Good luck, and have fun.

1. Let n be a positive even integer, and let $c_1, c_2, \ldots, c_{n-1}$ be real numbers satisfying

$$\sum_{i=1}^{n-1} |c_i - 1| < 1.$$

   Prove that

$$2x^n - c_{n-1}x^{n-1} + c_{n-2}x^{n-2} - \cdots - c_1x^1 + 2$$

   has no real roots.

2. Let $ABCD$ be a cyclic quadrilateral, and let E, F, G, and H be the midpoints of AB, BC, CD, and DA, respectively. Let W, X, Y, and Z be the orthocenters of triangles AHE, BEF, CFG, and DGH, respectively. Prove that quadrilaterals $ABCD$ and $WXYZ$ have the same area.

3. For a prime p, a subset S of residues modulo p is called a *sum-free multiplicative subgroup of* $\mathbb{F}_p$ if:

   - there is a nonzero residue α modulo p such that $S = \{1, \alpha^1, \alpha^2, \ldots\}$ (all considered mod p), and
   - there are no $a, b, c \in S$ (not necessarily distinct) such that $a + b \equiv c \pmod{p}$.

   Prove that for every integer N, there is a prime p and a sum-free multiplicative subgroup S of $\mathbb{F}_p$ such that $|S| \geq N$.

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments: straightedge, compass, set square (a.k.a., right triangle). Failure to meet any of these requirements will result in a 1-point automatic deduction.