Team Selection Test for the 54$^{\text{th}}$ IMO

December 13, 2012

1. A social club has $2k+1$ members, each of whom is fluent in the same k languages. Any pair of members always talk to each other in only one language. Suppose that there were no three members such that they use only one language among them. Let A be the number of three-member subsets such that the three distinct pairs among them use different languages. Find the maximum possible value of A.

2. Find all triples (x, y, z) of positive integers such that $x \le y \le z$ and
$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

3. Let ABC be a scalene triangle with $\angle BCA = 90^\circ$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. The circumcircle of triangle DKL intersects segment AB at a second point T (other than D). Prove that $\angle ACT = \angle BCT$.

4. Let f be a function from positive integers to positive integers, and let f^m be f applied m times. Suppose that for every positive integer n there exists a positive integer k such that $f^{2k}(n) = n + k$, and let k_n be the smallest such k. Prove that the sequence $k_1, k_2, \ldots$ is unbounded.

Team Selection Test for the 54$^{\text{th}}$ IMO

January 31, 2013

1. Two incongruent triangles ABC and XYZ are called a pair of *pals* if they satisfy the following conditions:

   (a) the two triangles have the same area;

   (b) let M and W be the respective midpoints of sides BC and YZ. The two sets of lengths $\{AB, AM, AC\}$ and $\{XY, XW, XZ\}$ are identical 3-element sets of pairwise relatively prime integers.

   Determine if there are infinitely many pairs of triangles that are pals of each other.

2. Let ABC be an acute triangle. Circle ω_1, with diameter AC, intersects side BC at F (other than C). Circle ω_2, with diameter BC, intersects side AC at E (other than C). Ray AF intersects ω_2 at K and M with $AK < AM$. Ray BE intersects ω_1 at L and N with $BL < BN$. Prove that lines AB, ML, NK are concurrent.

3. In a table with n rows and $2n$ columns where n is a fixed positive integer, we write either zero or one into each cell so that each row has n zeros and n ones. For $1 \le k \le n$ and $1 \le i \le n$, we define $a_{k,i}$ so that the i^{th} zero in the k^{th} row is the $a_{k,i}^{\text{th}}$ column. Let $\mathcal{F}$ be the set of such tables with $a_{1,i} \ge a_{2,i} \ge \cdots \ge a_{n,i}$ for every i with $1 \le i \le n$. We associate another $n \times 2n$ table $f(C)$ from $C \in \mathcal{F}$ as follows: for the k^{th} row of $f(C)$, we write n ones in the columns $a_{n,k} - k + 1, a_{n-1,k} - k + 2, \ldots, a_{1,k} - k + n$ (and we write zeros in the other cells in the row).

   (a) Show that $f(C) \in \mathcal{F}$.

   (b) Show that $f(f(f(f(f(f(C)))))) = C$ for any $C \in \mathcal{F}$.

4. Determine if there exists a (three-variable) polynomial $P(x, y, z)$ with integer coefficients satisfying the following property: a positive integer n is *not* a perfect square if and only if there is a triple (x, y, z) of positive integers such that $P(x, y, z) = n$.