Solutions to USA(J)MO 2016

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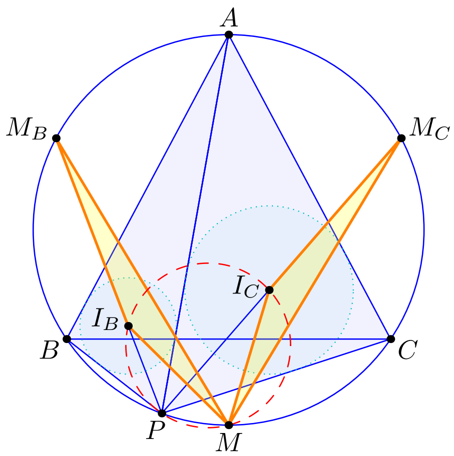
§1 Solution to JMO1

This problem was proposed by Ivan Borsenco and Zuming Feng.

Let M be the midpoint of arc BC not containing A. We claim M is the desired fixed point.



Since $\angle MPA = 90^\circ$ and ray PA bisects $\angle I_B P I_C$, it suffices to show that $MI_B = MI_C$. Let M_B, M_C be the second intersections of PI_B and PI_C with circumcircle. Now $M_B I_B = M_B B = M_C C = M_C I_C$, and moreover $MM_B = MM_C$, and $\angle I_B M_B M = \frac{1}{2}\widehat{PM} = \angle I_C M_C M$, so triangles $\triangle I_B M_B M \cong \triangle I_C M_C M$.

§2 Solution to JMO2

This problem was proposed by Evan Chen.

One answer is $n = 20 + 2^{19} = 524308$.

First, observe that

$$5^n \equiv 5^{20} \pmod{5^{20}}$$
$$5^n \equiv 5^{20} \pmod{2^{20}}$$

the former being immediate and the latter since $\varphi(2^{20}) = 2^{19}$. Hence $5^n \equiv 5^{20} \pmod{10^{20}}$. Moreover, we have

$$5^{20} = \frac{1}{2^{20}} \cdot 10^{20} < \frac{1}{1000^2} \cdot 10^{20} = 10^{-6} \cdot 10^{20}.$$

Thus the last 20 digits of 5^n will begin with six zeros.

§3 Solution to JMO3 / USAMO1

This problem was proposed by Iurie Boreico.

Solution with Danielle Wang: the answer is that $|S| \geq 8$.

Since we must have $2^{|S|} \geq 100$, we must have $|S| \geq 7$. We will provide an inductive construction for a *chain* of subsets $X_1, X_2, \dots, X_{2^{n-1}+1}$ of $S = \{1, \dots, n\}$ satisfying $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$ for each $n \geq 4$.

For $S = \{1, 2, 3, 4\}$, the following chain of length $2^3 + 1 = 9$ will work:

$$\{3,4\} \quad \{1\} \quad \{2,3\} \quad \{4\} \quad \{1,2\} \quad \{3\} \quad \{1,4\} \quad \{2\} \quad \{1,3\} \ .$$

Now, given a chain of subsets of $\{1, 2, \dots, n\}$ the following procedure produces a chain of subsets of $\{1, 2, \dots, n+1\}$:

1. take the original chain, delete any element, and make two copies of this chain, which now has even length;

2. glue the two copies together, joined by $\varnothing$ in between; and then

3. insert the element $n+1$ into the sets in alternating positions of the chain starting with the first.

For example, the first iteration of this construction gives:

345	1	235	4	125	3	145	2	5
34	15	23	45	12	35	14	25	

It can be easily checked that if the original chain satisfies the requirements, then so does the new chain, and if the original chain has length $2^{n-1}+1$, then the new chain has length $2^n + 1$, as desired. This construction yields a chain of length 129 when $S = \{1, 2, \dots, 8\}$.

To see that $|S| = 8$ is the minimum possible size, consider a chain on the set $S = \{1, 2, \dots, 7\}$ satisfying $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$. Because of these requirements any subset of size 4 or more can only be neighbored by sets of size 2 or less, of which there are $\binom{7}{1} + \binom{7}{2} = 28$ available. Thus, the chain can contain no more than 29 sets of size 4 or more and no more than 28 sets of size 2 or less. Finally, since there are only $\binom{7}{3} = 35$ sets of size 3 available, the total number of sets in such a chain can be at most $29 + 28 + 35 = 92 < 100$.

§4 Solution to USAMO2

This problem was proposed by Kiran Kedlaya.

We show the exponent of any given prime p is nonnegative in the expression. Recall that the exponent of p in $n!$ is equal to $\sum_{i \geq 1} \lfloor n/p^i \rfloor$. In light of this, it suffices to show that for any prime power P, we have

$$\left\lfloor \frac{k^2}{P} \right\rfloor \geq \sum_{j=0}^{k-1} \left(\left\lfloor \frac{j+k}{P} \right\rfloor - \left\lfloor \frac{j}{P} \right\rfloor \right).$$

Since both sides are integers, we show

$$\left\lfloor \frac{k^2}{P} \right\rfloor > -1 + \sum_{j=0}^{k-1} \left(\left\lfloor \frac{j+k}{P} \right\rfloor - \left\lfloor \frac{j}{P} \right\rfloor \right).$$

If we denote by $\{x\}$ the fractional part of x, then $\lfloor x \rfloor = x - \{x\}$, and so the previous inequality can be rewritten

$$\left\{\frac{k^2}{P}\right\} + \sum_{j=0}^{k-1} \left\{\frac{j}{P}\right\} < 1 + \sum_{j=0}^{k-1} \left\{\frac{j+k}{P}\right\}$$

However, the sum of remainders when $0, 1, \dots, k-1$ are taken modulo P is easily seen to be less than the sum of remainders when $k, k+1, \dots, 2k-1$ are taken modulo P. So

$$\sum_{j=0}^{k-1} \left\{\frac{j}{P}\right\} \le \sum_{j=0}^{k-1} \left\{\frac{j+k}{P}\right\}$$

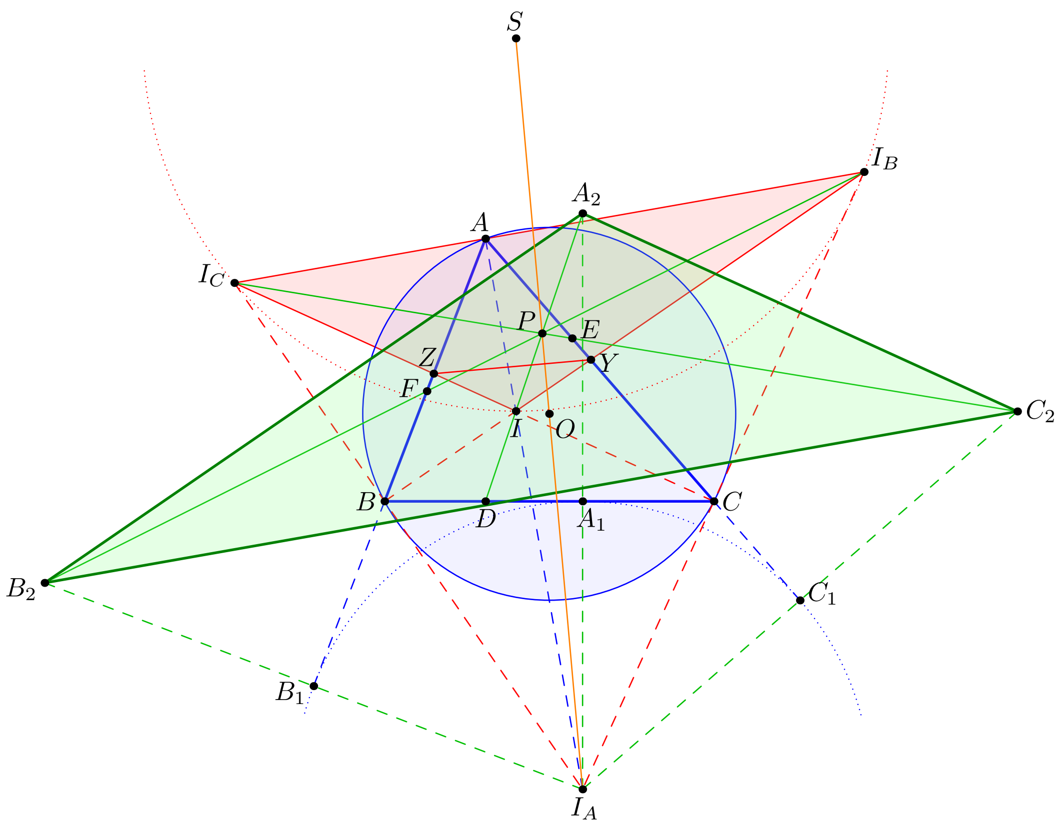
follows, and we are done upon noting $\left\{k^2/P\right\} < 1$.

§5 Solution to USAMO3

This problem was proposed by Evan Chen.

This problem was jointly written by me and Telv Cohl.

Let I_A denote the A-excenter and I the incenter. Then let D denote the foot of the altitude from A. Suppose the A-excircle is tangent to $\overline{BC}$, $\overline{AB}$, $\overline{AC}$ at A_1, B_1, C_1 and let A_2, B_2, C_2 denote the reflections of I_A across these points. Let S denote the circumcenter of $\triangle I I_B I_C$.



We begin with the following observation: points D, I, A_2 are collinear, as are points E, I_C, C_2 are collinear and points F, I_B, B_2 are collinear. This follows from the "midpoints of altitudes" lemma.

Observe that $\overline{B_2C_2} \parallel \overline{B_1C_1} \parallel \overline{I_BI_C}$. Proceeding similarly on the other sides, we discover $\triangle II_BI_C$ and $\triangle A_2B_2C_2$ are homothetic. Hence P is the center of this homothety (in particular, D, I, P, A_2 are collinear). Moreover, P lies on the line joining I_A to S, which is the Euler line of $\triangle II_BI_C$, so it passes through the nine-point center of $\triangle II_BI_C$, which is O. Consequently, P, O, I_A are collinear as well.

To finish, we need only prove that $\overline{OS} \perp \overline{YZ}$. In fact, we claim that $\overline{YZ}$ is the radical axis of the circumcircles of $\triangle ABC$ and $\triangle II_BI_C$. Actually, Y is the radical center of these two circumcircles and the circle with diameter $\overline{II_B}$ (which passes through A and C). Analogously Z is the radical center of the circumcircles and the circle with diameter $\overline{II_C}$, and the proof is complete.

§6 Solution to JMO4

This problem was proposed by Gregory Galperin.

The answer is

$$N = 2017 + 2018 + \cdots + 4032 = 1008 \cdot 6049 = 6097392.$$

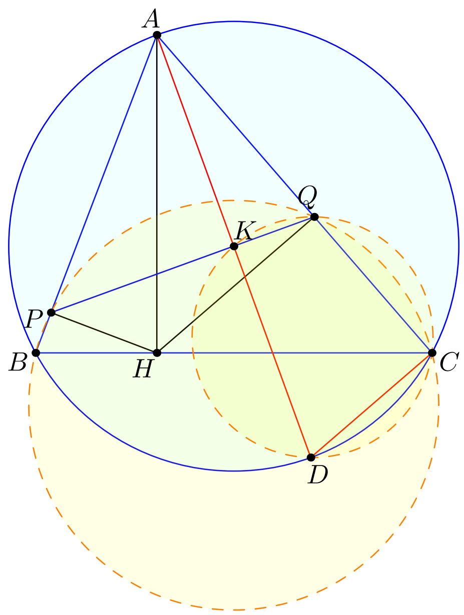
To see that N must be at least this large, consider the situation when 1, 2, ..., 2016 are removed. Among the remaining elements, any sum of 2016 elements is certainly at least $2017 + 2018 + \cdots + 4032$.

Now we show this value of N works. Consider the 3024 pairs of numbers $(1, 6048)$, $(2, 6047)$, ..., $(3024, 3025)$. Regardless of which 2016 elements of $\{1, 2, \dots, N\}$ are deleted, at least $3024 - 2016 = 1008$ of these pairs have both elements remaining. Since each pair has sum 6049, we can take these pairs to be the desired numbers.

§7 Solution to JMO5

This problem was proposed by Jacek Fabrykowski and Zuming Feng.

First, since $AP \cdot AB = AH^2 = AQ \cdot AC$, it follows that $PQCB$ is cyclic. Consequently, we have $AO \perp PQ$.



Let K be the foot of A onto PQ, and let D be the point diametrically opposite A. Thus A, K, O, D are collinear.

Since quadrilateral $KQCD$ is cyclic ($\angle QKD = \angle QCD = 90^\circ$), we have

$$AK \cdot AD = AQ \cdot AC = AH^2 \implies AK = \frac{AH^2}{AD} = \frac{AH^2}{2AO} = AO$$

so $K = O$.

§8 Solution to JMO6 / USAMO4

This problem was proposed by Titu Andreescu.

First, taking $x = y = 0$ in the given yields $f(0) = 0$, and then taking $x = 0$ gives $f(y)f(-y) = f(y)^2$. So also $f(-y)^2 = f(y)f(-y)$, from which we conclude f is even. Then taking $x = -y$ gives

$$\forall x \in \mathbb{R}: \qquad f(x) = x^2 \qquad \text{or} \qquad f(4x) = 0 \qquad (\star)$$

for all x.

Next, we claim that

$$\forall x \in \mathbb{R}: \qquad f(x) = x^2 \qquad \text{or} \qquad f(x) = 0 \qquad (\heartsuit)$$

To see this assume $f(t) \neq 0$ (hence $t \neq 0$). By $(\star)$ we get $f(t/4) = t^2/16$. Now take $(x, y) = (3t/4, t/4)$ to get

$$\frac{t^2}{4} f(2t) = f(t)^2 \implies f(2t) \neq 0.$$

If we apply $(\star)$ again we actually also get $f(t/2) \neq 0$. Together these imply

$$f(t) \neq 0 \iff f(2t) \neq 0 \qquad (\spadesuit).$$

Repeat $(\spadesuit)$ to get $f(4t) \neq 0$, hence $f(t) = t^2$, proving $(\heartsuit)$.

We are now ready to prove the claim that the only two functions satisfying the requirements are $f(x) = 0$ for all $x \in \mathbb{R}$ and $f(x) = x^2$ for all $x \in \mathbb{R}$.

Assume there's an $a \neq 0$ for which $f(a) = 0$; we show that $f \equiv 0$.

Let $b \in \mathbb{R}$ be given. Since f is even, we can assume without loss of generality that $a, b > 0$. Also, note that $f(x) \geq 0$ for all x by $(\heartsuit)$. By using $(\spadesuit)$ we can generate $c > b$ such that $f(c) = 0$ by taking $c = 2^n a$ for a large enough integer n. Now, select $x, y > 0$ such that $x - 3y = b$ and $x + y = c$. That is,

$$(x, y) = \left(\frac{3c + b}{4}, \frac{c - b}{4} \right).$$

Substitution into the original equation gives

$$0 = (f(x) + xy)\, f(b) + (f(y) + xy)\, f(3x - y) = (f(x) + f(y) + 2xy)\, f(b)$$

Since $f(x) + f(y) + 2xy > 0$, if follows that $f(b) = 0$, as desired.

§9 Solution to USAMO5

This problem was proposed by Ivan Borsenco.

Here is a very short solution with complex numbers, from the official solutions file. In fact, it shows that we only need $AM = AQ = NP$ and $MN = QP$.

We use complex numbers with ABC the unit circle, assuming WLOG that A, B, C are labeled counterclockwise. Let x, y, z be the complex numbers corresponding to the arc midpoints of BC, CA, AB, respectively; thus $x + y + z$ is the incenter of $\triangle ABC$. Finally, let $s > 0$ be the side length of $AM = AQ = NP$.

Then, since $MA = s$ and $MA \perp OX$, it follows that

$$m - a = i \cdot sx.$$

Similarly, $n - p = i \cdot sy$ and $a - q = i \cdot sz$, so summing these up gives

$$i \cdot s(x + y + z) = (p - q) + (m - n) = (m - n) - (q - p).$$

Since $MN = PQ$, the argument of $(m - n) - (q - p)$ is along the external angle bisector of the angle formed, which is perpendicular to ℓ. On the other hand, $x + y + z$ is oriented in the same direction as OI, as desired.

§10 Solution to USAMO6

This problem was proposed by Gabriel Carroll.

The game is winnable if and only if $k < n$.

First, suppose $2 \le k < n$. Query the cards in positions $\{1, \dots, k\}$, then $\{2, \dots, k + 1\}$, and so on, up to $\{2n - k + 1, 2n\}$. By taking the difference of any two adjacent queries, we can deduce for certain the values on cards $1, 2, \dots, 2n - k$. If $k \le n$, this is more than n cards, so we can find a matching pair.

For $k = n$ we remark the following: at each turn after the first, assuming one has not won, there are n cards representing each of the n values exactly once, such that the player has no information about the order of those n cards. We claim that consequently the player cannot guarantee victory. Indeed, let S denote this set of n cards, and $\overline{S}$ the other n cards. The player will never win by picking only cards in S or $\overline{S}$. Also, if the player selects some cards in S and some cards in $\overline{S}$, then it is possible that the choice of cards in S is exactly the complement of those selected from $\overline{S}$; the strategy cannot prevent this since the player has no information on S. This implies the result.