# IMO 2018 Solution Notes

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This is an unofficial solutions packet for the 2018 IMO. In general, they are a combination of my own work, as well as the official solutions provided by the organizers (for which they hold any copyrights), and solutions found on the Art of Problem Solving forums.

Corrections and comments are welcome!

## Contents

# §0 Problems

1. Let $\Gamma$ be the circumcircle of acute triangle $ABC$. Points $D$ and $E$ lie on segments $AB$ and $AC$, respectively, such that $AD = AE$. The perpendicular bisectors of $\overline{BD}$ and $\overline{CE}$ intersect the minor arcs $AB$ and $AC$ of $\Gamma$ at points $F$ and $G$, respectively. Prove that the lines $DE$ and $FG$ are parallel.

2. Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \dots, a_n$ satisfying

$$a_i a_{i+1} + 1 = a_{i+2}$$

for $i = 1, 2, \dots, n$, where indices are taken modulo $n$.

3. An *anti-Pascal triangle* is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following array is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to $1 + 2 + \dots + 2018$?

4. A *site* is any point $(x, y)$ in the plane for which $x, y \in \{1, \dots, 20\}$. Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones on unoccupied sites, with Amy going first; Amy has the additional restriction that no two of her stones may be at a distance equal to $\sqrt{5}$. They stop once either player cannot move. Find the greatest $K$ such that Amy can ensure that she places at least $K$ stones.

5. Let $a_1, a_2, \dots$ be an infinite sequence of positive integers. Suppose that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \dots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer for all sufficiently large $n$. Prove that $(a_n)$ is eventually constant.

6. A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point $X$ lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$

Prove that $\angle BXA + \angle DXC = 180^\circ$.

# §1 IMO 2018/1, proposed by HEL

Let $\Gamma$ be the circumcircle of acute triangle $ABC$. Points $D$ and $E$ lie on segments $AB$ and $AC$, respectively, such that $AD = AE$. The perpendicular bisectors of $\overline{BD}$ and $\overline{CE}$ intersect the minor arcs $AB$ and $AC$ of $\Gamma$ at points $F$ and $G$, respectively. Prove that the lines $DE$ and $FG$ are parallel.

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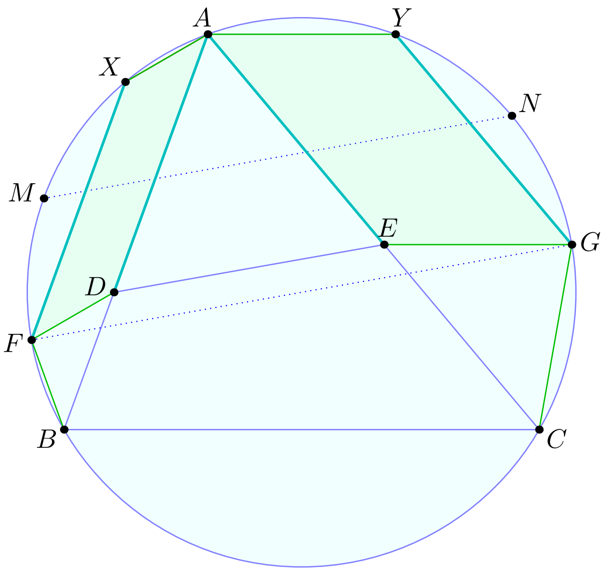
Construct parallelograms $AXFD$ and $AEGY$, noting that $X$ and $Y$ lie on $\Gamma$. As $\overline{XF} \parallel \overline{AB}$ we can let $M$ denote the midpoint of minor arcs $\widehat{XF}$ and $\widehat{AB}$. Define $N$ similarly.




Observe that $XF = AD = AE = YG$, so arcs $\widehat{XF}$ and $\widehat{YG}$ have equal measure; hence arcs $\widehat{MF}$ and $\widehat{NG}$ have equal measure; therefore $\overline{MN} \parallel \overline{FG}$.

Since $\overline{MN}$ and $\overline{DE}$ are both perpendicular to the $\angle A$ bisector, so we're done.

# §2 IMO 2018/2, proposed by SVK

Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \dots, a_n$ satisfying

$$a_i a_{i+1} + 1 = a_{i+2}$$

for $i = 1, 2, \dots, n$, where indices are taken modulo $n$.

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The answer is $3 \mid n$, achieved by $(-1, -1, 2, -1, -1, 2, \dots)$. We present two solutions.

**First solution by inequalities** We compute $a_i a_{i+1} a_{i+2}$ in two ways:

$$\begin{aligned} a_i a_{i+1} a_{i+2} &= [a_{i+2} - 1] a_{i+2} = a_{i+2}^2 - a_{i+2} \\ &= a_i [a_{i+3} - 1] = a_i a_{i+3} - a_i. \end{aligned}$$

Cyclically summing $a_{i+2}^2 - a_{i+2} = a_i a_{i+3} - a_i$ then gives

$$\sum_i a_{i+2}^2 = \sum_i a_i a_{i+3} \iff \sum_{\text{cyc}} (a_i - a_{i+3})^2 = 0.$$

This means for inequality reasons the sequence is 3-periodic. Since the sequence is clearly not 1-periodic, as $x^2 + 1 = x$ has no real solutions. Thus $3 \mid n$.

**Second solution by sign counting** Extend $a_n$ to be a periodic sequence. The idea is to look at the signs, and show the sequence of the signs must be $--+$ repeated. This takes several steps:

- The pattern $---$ is impossible. Obvious, since the third term should be $> 1$.
- The pattern $++$ is impossible. Then the sequence becomes strictly increasing, hence may not be periodic.
- Zeros are impossible. If $a_1 = 0$, then $a_2 = 0$, $a_3 > 0$, $a_4 > 0$, which gives the impossible $++$.
- The pattern $--+-+$ is impossible. Compute the terms:

  $$\begin{aligned} a_1 &= -x < 0 \\ a_2 &= -y < 0 \\ a_3 &= 1 + xy > 1 \\ a_4 &= 1 - y(1 + xy) < 0 \\ a_5 &= 1 + (1 + xy)(1 - y(1 + xy)) < 1. \end{aligned}$$

  But now

  $$a_6 - a_5 = (1 + a_5 a_4) - (1 + a_3 a_4) = a_4(a_5 - a_3) > 0$$

  since $a_5 > 1 > a_3$. This means we have the impossible $++$ pattern.
- The infinite alternating pattern $-+-+-+-+\dots$ is impossible. Note that

  $$a_1 a_2 + 1 = a_3 < 0 < a_4 = 1 + a_2 a_3 \implies a_1 < a_3$$

  since $a_2 > 0$; extending this we get $a_1 < a_3 < a_5 < \dots$ which contradicts the periodicity.

We finally collate the logic of sign patterns. Since the pattern is not alternating, there must be $--$ somewhere. Afterwards must be $+$, and then after that must be two minus signs (since one minus sign is impossible by impossibility of $--+-+$ and $---$ is also forbidden); thus we get the periodic $--+$ as desired.

# §3 IMO 2018/3, proposed by IRN

An *anti-Pascal triangle* is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following array is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to $1 + 2 + \cdots + 2018$?

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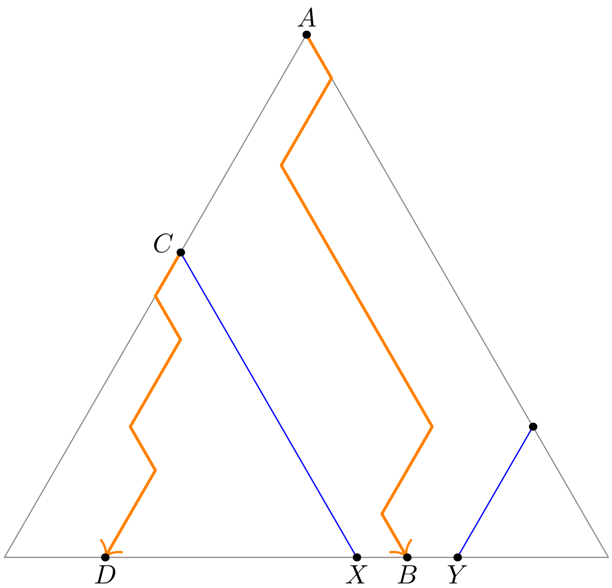
Let $n = 2018$ and $N = 1 + 2 + \cdots + n$. For every number $d$ not in the bottom row, draw an arrow from $d$ to the larger of the two numbers below it (i.e. if $d = a - b$, draw $d \to a$). This creates an *oriented forest* (which looks like lightning strikes).

Consider the directed path starting from the top vertex $A$. Starting from the first number, it increments by at least $1 + 2 + \cdots + n$, since the increments at each step in the path are distinct; therefore equality must hold and thus the path from the top ends at $N = 1 + 2 + \cdots + n$ with all the numbers $\{1, 2, \dots, n\}$ being close by. Let $B$ be that position.




Consider the two left/right neighbors $X$ and $Y$ of the endpoint $B$. Assume that $B$ is to the right of the midpoint of the bottom side, and complete the equilateral triangle as shown to an apex $C$. Consider the lightning strike from $C$ hitting the bottom at $D$. It travels at least $\lfloor n/2 - 1 \rfloor$ steps, by construction. But the increases must be at least $n + 1$, $n + 2$, ... since $1, 2, \dots, n$ are close to the $A \to B$ lightning path. Then the number at $D$ is at least

$$(n + 1) + (n + 2) + \cdots + (n + (\lfloor n/2 - 1 \rfloor)) > 1 + 2 + \cdots + n$$

for $n \geq 2018$, contradiction.

# §4 IMO 2018/4, proposed by ARM

A *site* is any point $(x, y)$ in the plane for which $x, y \in \{1, \ldots, 20\}$. Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones on unoccupied sites, with Amy going first; Amy has the additional restriction that no two of her stones may be at a distance equal to $\sqrt{5}$. They stop once either player cannot move. Find the greatest $K$ such that Amy can ensure that she places at least $K$ stones.

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The answer is $K = 100$.

First, we show Amy can always place at least 100 stones. Indeed, treat the problem as a grid with checkerboard coloring. Then Amy can choose to always play on one of the 200 black squares. In this way, she can guarantee half the black squares, i.e. she can get $\frac{1}{2} \cdot 200 = 100$ stones.

Second, we show Ben can prevent Amy from placing more than 100 stones. Divide into several $4 \times 4$ squares and then further partition each $4 \times 4$ squares as shown in the grid below.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

The squares with each label form 4-cycles by knight jumps. For each such cycle, whenever Amy plays in the cycle, Ben plays in the opposite point of the cycle, preventing Amy from playing any more stones in that original cycle. Hence Amy can play at most in $1/4$ of the stones, as desired.

## §5 IMO 2018/5, proposed by MNG

Let $a_1, a_2, \dots$ be an infinite sequence of positive integers. Suppose that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \dots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer for all sufficiently large $n$. Prove that $(a_n)$ is eventually constant.

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The condition implies that the difference $S(n) = \frac{a_{n+1}}{a_1} - \frac{a_n}{a_1} + \frac{a_n}{a_{n+1}}$ is an integer for all $n > N$. We proceed by $p$-adic valuation only henceforth.

**Claim** — If $p \nmid a_1$, then $\nu_p(a_{n+1}) \le \nu_p(a_n)$ for $n \ge N$.

*Proof.* The first two terms of $S(n)$ have nonnegative $\nu_p$, so we need $\nu_p(\frac{a_n}{a_{n+1}}) \ge 0$. □

**Claim** — If $p \mid a_1$, then $\nu_p(a_n)$ is eventually constant.

*Proof.* By hypothesis $\nu_p(a_1) > 0$. We consider two cases.

- First assume $\nu_p(a_k) \ge \nu_p(a_1)$ for some $k > N$. We claim that for any $n \ge k$ we have:
$$\nu_p(a_1) \le \nu_p(a_{n+1}) \le \nu_p(a_n).$$
This is just by induction on $n$; from $\nu(\frac{a_n}{a_1}) \ge 0$, we have
$$\nu_p\left(\frac{a_{n+1}}{a_1} + \frac{a_n}{a_{n+1}}\right) \ge 0$$
which implies the displayed inequality (since otherwise exactly one term of $S(n)$ has nonnegative $\nu_p$). Thus once we reach this case, $\nu_p(a_n)$ is monotic but bounded below by $\nu_p(a_1)$, and so it is eventually constant.

- Now assume $\nu_p(a_k) < \nu_p(a_1)$ for every $k > N$. Take any $n > N$ then. We have $\nu_p\left(\frac{a_{n+1}}{a_1}\right) < 0$, and also $\nu_p\left(\frac{a_n}{a_1}\right) < 0$, so among the three terms of $S(n)$, two must have equal $p$-adic valuation. We consider all three possibilities:
$$\nu_p\left(\frac{a_{n+1}}{a_1}\right) = \nu_p\left(\frac{a_n}{a_1}\right) \implies \boxed{\nu_p(a_{n+1}) = \nu_p(a_n)}$$
$$\nu_p\left(\frac{a_{n+1}}{a_1}\right) = \nu_p\left(\frac{a_n}{a_{n+1}}\right) \implies \boxed{\nu_p(a_{n+1}) = \frac{\nu_p(a_n) + \nu_p(a_1)}{2}}$$
$$\nu_p\left(\frac{a_n}{a_1}\right) = \nu_p\left(\frac{a_n}{a_{n+1}}\right) \implies \nu_p(a_{n+1}) = \nu_p(a_1), \text{ but this is impossible.}$$
Thus, $\nu_p(a_{n+1}) \ge \nu_p(a_n)$ and $\nu_p(a_n)$ is bounded above by $\nu_p(a_1)$. So in this case we must also stabilize. □

Since the latter claim is applied only to finitely many primes, after some time $\nu_p(a_n)$ is fixed for all $p \mid a_1$. Afterwards, the sequence satisfies $a_{n+1} \mid a_n$ for each $n$, and thus must be eventually constant.

**Remark.** This solution is almost completely $p$-adic, in the sense that I think a similar result if one replaces $a_n \in \mathbb{Z}$ by $a_n \in \mathbb{Z}_p$ for any particular prime $p$. In other words, the primes almost do not talk to each other.

There is one caveat: if $x_n$ is an integer sequence such that $\nu_p(x_n)$ is eventually constant for each prime then $x_n$ may not be constant. For example, take $x_n$ to be the $n$th prime! That's why in the first claim (applied to co-finitely many of the primes), we need the stronger non-decreasing condition, rather than just eventually constant.

## §6 IMO 2018/6, proposed by POL

A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point $X$ lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$

Prove that $\angle BXA + \angle DXC = 180°$.

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We present two solutions by inversion. The first is the official one. The second is a solution via inversion, completed by USA5 Michael Ren.

**Official solution by inversion** In what follows a convex quadrilateral is called *quasi-harmonic* if $AB \cdot CD = BC \cdot CA$.

> **Claim** — A quasi-harmonic quadrilateral is determined up to similarity by its angles.

(This could be expected by degrees of freedom; a quadrilateral has four degrees of freedom up to similarity; the pseudo-harmonic condition is one while the angles provide three conditions.)

*Proof.* Do some inequalities. □

Performing an inversion at $X$, one obtains a second quasi-harmonic quadrilateral $A^*B^*C^*D^*$ which has the same angles as the original one, $\angle D^* = \angle A$, $\angle A^* = \angle B$, and so on. Thus by the claim we obtain similarity

$$D^*A^*B^*C^* \sim ABCD.$$

If one then maps $D^*A^*B^*C^*$, onto $ABCD$, the image of $X^*$ becomes a point isogonally conjugate to $X$. In other words, $X$ has an isogonal conjugate in $ABCD$.

It is well-known that this is equivalent to $\angle BXA + \angle DXC = 180°$, for example by inscribing an ellipse with foci $X$ and $X^*$.

**Second solution: "rhombus inversion", by Michael Ren** Since

$$\frac{AB}{AD} = \frac{CB}{CD}$$

and

$$\frac{BA}{BC} = \frac{DA}{DC}$$

it follows that $B$ and $D$ lie on an Apollonian circle $\omega_{AC}$ through $A$ and $C$, while $A$ and $C$ lie on an Apollonian circle $\omega_{BD}$ through $B$ and $D$. We let these two circles intersect at a point $P$ inside $ABCD$.

The main idea is then to perform an inversion about $P$ with radius 1. We obtain:

> **Lemma**
> The image of $ABCD$ is a rhombus.

*Proof.* By the inversion distance formula, we have

$$\frac{1}{A'B'} = \frac{PA}{AB} \cdot PB = \frac{PC}{BC} \cdot PB = \frac{1}{B'C'}$$

and so $A'B' = B'C'$. In a similar way, we derive $B'C' = C'D' = D'A'$, so the image is a rhombus as claimed. □

Let us now translate the angle conditions. We were given that $\measuredangle XAB = \measuredangle XCD$, but

$$\begin{aligned}\measuredangle XAB &= \measuredangle XAP + \measuredangle PAB = \measuredangle PX'A' + \measuredangle A'B'P \\ \measuredangle XCD &= \measuredangle XCP + \measuredangle PCD = \measuredangle PX'C' + \measuredangle C'D'P\end{aligned}$$

so subtracting these gives

$$\begin{aligned}\measuredangle A'X'C' &= \measuredangle A'B'P + \measuredangle PD'C' = \measuredangle(A'B', B'P) + \measuredangle(PD', C'D') \\ &= \measuredangle(A'B', B'P) + \measuredangle(PD', A'B') = \measuredangle D'PB'. \end{aligned} \tag{1}$$

since $\overline{A'B'} \parallel \overline{C'D'}$. Similarly, we obtain

$$\measuredangle B'X'D' = \measuredangle A'PC'. \tag{2}$$

We now translate the desired condition. Since

$$\begin{aligned}\measuredangle AXB &= \measuredangle AXP + \measuredangle PXB = \measuredangle PA'X' + \measuredangle X'B'P \\ \measuredangle CXD &= \measuredangle CXP + \measuredangle PXD = \measuredangle PC'X' + \measuredangle X'DP'\end{aligned}$$

we compute

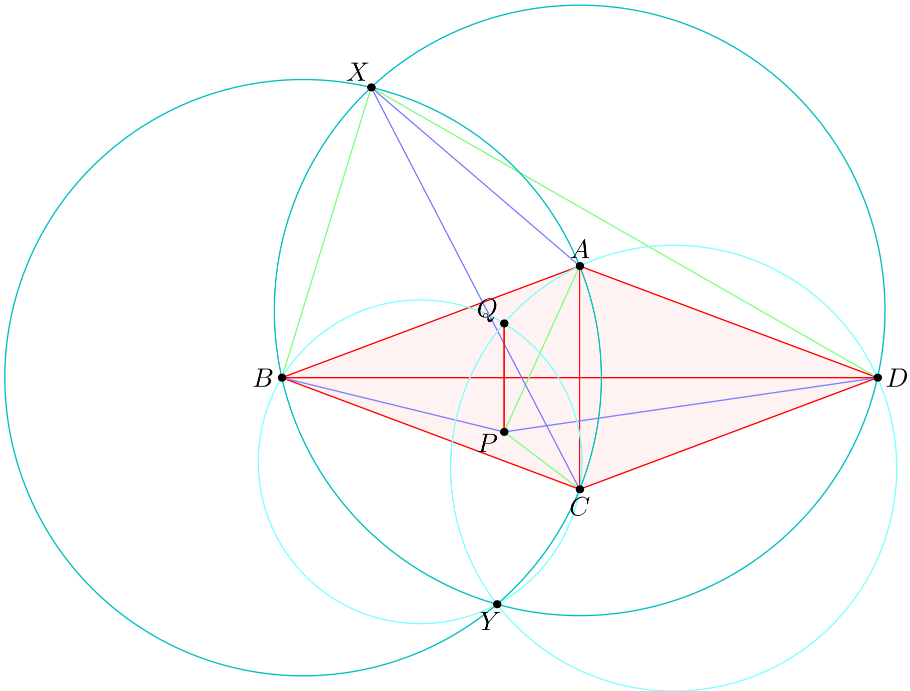
$$\begin{aligned}\measuredangle AXB + \measuredangle CXD &= (\measuredangle PA'X' + \measuredangle X'B'P) + (\measuredangle PC'X' + \measuredangle X'D'P) \\ &= -\left[\left(\measuredangle A'X'P + \measuredangle X'PA'\right) + \left(\measuredangle PX'B' + \measuredangle B'PX'\right)\right] \\ &\quad - \left[\left(\measuredangle C'X'P + \measuredangle X'PC'\right) + \left(\measuredangle PX'D' + \measuredangle D'PX'\right)\right] \\ &= \left[\measuredangle PX'A' + \measuredangle BX'P + \measuredangle PX'C' + \measuredangle D'X'P\right] \\ &\quad + \left[\measuredangle A'PX' + \measuredangle X'PB' + \measuredangle C'PX' + \measuredangle X'PD'\right] \\ &= \measuredangle A'PB' + \measuredangle C'PD' + \measuredangle B'X'C + \measuredangle D'X'A\end{aligned}$$

and we wish to show this is equal to zero, i.e. the desired becomes

$$\measuredangle A'PB' + \measuredangle C'PD' + \measuredangle B'X'C + \measuredangle D'X'A = 0. \tag{3}$$

In other words, the problem is to show (1) and (2) implies (3).

Henceforth drop apostrophes. Here is the inverted diagram (with apostrophes dropped).




Let $Q$ denote the reflection of $P$ and let $Y$ denote the second intersection of $(BQC)$ and $(AQD)$. Then

$$\begin{aligned}-\measuredangle AXC = -\measuredangle DPB = \measuredangle BQD = \measuredangle BQY + \measuredangle YQD = \measuredangle BCY + \measuredangle YAD \\ = \measuredangle(BC, CY) + \measuredangle(YA, AD) = \measuredangle YCA = -\measuredangle AYC.\end{aligned}$$

Hence $XACY$ is concyclic; similarly $XBDY$ is concyclic.

**Claim** — $X \neq Y$.

*Proof.* To see this: Work pre-inversion assuming $AB < AC$. Then $Q$ was the center of $\omega_{BD}$. If $T$ was the second intersection of $BA$ with $(QBC)$, then $QB = QD = QT = \sqrt{QA \cdot QC}$, by shooting lemma. Since $\angle BAD < 180°$, it follows $(QBCY)$ encloses $ABCD$ (pre-inversion). (This part is where the hypothesis that $ABCD$ is convex with $X$ inside is used.) □

Finally, we do an angle chase to finish:

$$\begin{aligned}\measuredangle DXA &= \measuredangle DXY + \measuredangle YXA = \measuredangle DBY + \measuredangle YCA \\ &= \measuredangle(DB, YB) + \measuredangle(CY, CA) = \measuredangle CYB + 90^\circ \\ &= \measuredangle CQB + 90^\circ = -\measuredangle APB + 90^\circ. \qquad (4)\end{aligned}$$

Similarly,

$$\measuredangle BXC = \measuredangle DPC + 90^\circ. \qquad (5)$$

Summing (4) and (5) gives (3).

**Remark.** A difficult part of the problem in many solutions is that the conclusion is false in the directed sense, if the point $X$ is allowed to lie outside the quadrilateral. We are saved in the first solution because the equivalence of the isogonal conjugation requires $X$ inside the quadrilateral. On the other hand, in the second solution, the issue appears in he presence of the second point $Y$.

**Third solution by moving points (Anant Mudgal, un-edited)** Let $P = \overline{AD} \cap \overline{BC}$ and $Q = \overline{AB} \cap \overline{CD}$. Let $x = \frac{1}{2}(\angle A - \angle C)$ and $y = \frac{1}{2}(\angle B - \angle D)$. WLOG $x, y \ge 0$. Now construct point $X$ inside $ABCD$ such that $\angle BXA = 90^\circ - y$ and $\angle BXC = 90^\circ - x$. Then $\angle AXC = 180^\circ - \frac{1}{2}(\angle A + \angle B - \angle C - \angle D) = \angle C + \angle D = 180^\circ - \angle APC$ hence $X \in \odot(APC)$.

**Claim** — $QBXD$ is cyclic.

*Proof.* Invert at $B$ with radius $\sqrt{BA \cdot BC}$ followed by reflection in internal bisector of $\angle ABC$. We omit the $'$s for readability. Thus, in $\triangle ABC$ with $M$ midpoint of side $\overline{AC}$ we have

- point $D$ satisfies $\overline{DM} \perp \overline{AC}$;
- point $Q \ne B$ lies on line $\overline{AB}$ and $\odot(BDC)$;
- point $X$ satisfies $\angle BAX = 90^\circ + \theta$ and $\angle BCX = 90^\circ - \angle B + \alpha$; where $\theta = \angle(\overline{BD}, \overline{DM})$ and $\alpha = \angle ADM$.

We need to prove $X$ lies on line $\overline{DQ}$. Define $T$ as the point with $\overline{TC} \perp \overline{BC}$ and $TA = TC$. Let $m$ be the line with $T \in m$ and $m \perp \overline{BA}$. Now define $X_1 \in m$ with $\angle BAX_1 = 90^\circ + \theta$ and $X_2 \in m$ with $\angle BCX_2 = 90^\circ - \angle B + \alpha$. We show that $X_1 \equiv X_2$.

Move point $D$ on the perpendicular bisector $\ell$ of $\overline{AC}$. Then pencils $\overline{AX_1}$ and $\overline{BD}$ move with equal angular velocity (not constant though). Hence $D \mapsto X_1$ is a projective map. Likewise since pencils $\overline{AD}$ and $\overline{CX_2}$ move with equal angular velocity (not constant though), $D \mapsto X_2$ is also a projective map. Thus there is a projective mapping $\pi : X_1 \mapsto X_2$. In order to show $\pi$ is the identity; we only need to show it has three fixed points.

- For $D = \ell_\infty$ note that $\alpha = -180^\circ$ and $\theta = 0^\circ$ so $\overline{AX_1} \perp \overline{AB}$ and $\overline{CX_2} \perp \overline{AB}$ hence $X_1 = X_2 = m_\infty$.
- For $D = m \cap \overline{BA}$ note that $\alpha = 90^\circ - \angle A$ so $\overline{CX_2} \equiv \overline{AC}$ and $\theta = 90^\circ + \angle A$ so $\overline{AX_1} \equiv \overline{AC}$.
- For $D = O$ (circumcenter of $\triangle ABC$) note that $\alpha = \angle B$ so $X_2 = T$. Since $TA = TC$ we obtain $\angle TAB = 90^\circ + \angle A - \angle C$ but $\theta = \angle A - \angle C$ so $X_1 = T$ as well.

Now $X = \overline{AX_1} \cap \overline{CX_2}$ hence $X \overset{\text{def}}{:=} X_1 \equiv X_2$. So $\angle CDT = \alpha$ and $\angle CXT = \angle B - (\angle B - \alpha)$ hence $X \in \odot(TCB)$. Finally, observe that $\angle XDC = \angle(\overline{CT}, m) = \angle B$ hence $\angle CDX + \angle CDQ = \angle CBA + \angle CBQ = 180^\circ$ so the claim is proved. Apply directed angles to conclude the same for other configurations. □

Then

$$X \in \odot(QBD) \implies \angle BXA - \angle CXD = \angle BXD - \angle AXC = \angle C + \angle D - \angle C - \angle B = \angle D - \angle B = -2x$$

and $\angle BXA = (90^\circ - x)$ so $\angle CXD = (90^\circ + x)$; hence $\angle BXA + \angle DXC = 180^\circ$. Now $PAXC$ and $QBXD$ are cyclic so $\angle XAD = \angle XCB$ and $\angle XDC = \angle XBA$; so we're done!