Team Selection Test for the 59$^{\text{th}}$ International Mathematical Olympiad

United States of America

Day I

Thursday, December 7, 2017

Time limit: 4.5 hours. Each problem is worth 7 points. You may keep the exam problems, but do not discuss them with anyone until Monday, December 11 at noon Eastern time.

IMO TST 1. Let $n \geq 2$ be a positive integer, and let $\sigma(n)$ denote the sum of the positive divisors of n. Prove that the n^{th} smallest positive integer relatively prime to n is at least $\sigma(n)$, and determine for which n equality holds.

IMO TST 2. Find all functions $f\colon \mathbb{Z}^2 \to [0,1]$ such that for any integers x and y,

$$f(x,y) = \frac{f(x-1,y) + f(x,y-1)}{2}.$$

IMO TST 3. At a university dinner, there are 2017 mathematicians who each order two distinct entrées, with no two mathematicians ordering the same pair of entrées. The cost of each entrée is equal to the number of mathematicians who ordered it, and the university pays for each mathematician's less expensive entrée (ties broken arbitrarily). Over all possible sets of orders, what is the maximum total amount the university could have paid?

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Day II

Thursday, January 18, 2018

Time limit: 4.5 hours. Each problem is worth 7 points. You may keep the exam problems, but do not discuss them with anyone until Monday, January 22 at noon Eastern time.

IMO TST 4. Let n be a positive integer and let $S \subseteq \{0,1\}^n$ be a set of binary strings of length n. Given an odd number $x_1, \dots, x_{2k+1} \in S$ of binary strings (not necessarily distinct), their *majority* is defined as the binary string $y \in \{0,1\}^n$ for which the i^{th} bit of y is the most common bit among the i^{th} bits of $x_1, \dots, x_{2k+1}$. (For example, if $n = 4$ the majority of 0000, 0000, 1101, 1100, 0101 is 0100.)

Suppose that for some positive integer k, S has the property P_k that the majority of any $2k+1$ binary strings in S (possibly with repetition) is also in S. Prove that S has the same property P_k for all positive integers k.

IMO TST 5. Let $ABCD$ be a convex cyclic quadrilateral which is not a kite, but whose diagonals are perpendicular and meet at H. Denote by M and N the midpoints of $\overline{BC}$ and $\overline{CD}$. Rays MH and NH meet $\overline{AD}$ and $\overline{AB}$ at S and T, respectively. Prove there exists a point E, lying outside quadrilateral $ABCD$, such that

- ray EH bisects both angles $\angle BES$, $\angle TED$, and
- $\angle BEN = \angle MED$.

IMO TST 6. Alice and Bob play a game. First, Alice secretly picks a finite set S of lattice points in the Cartesian plane. Then, for every line ℓ in the plane which is horizontal, vertical, or has slope $+1$ or -1, she tells Bob the number of points of S that lie on ℓ. Bob wins if he can then determine the set S.

Prove that if Alice picks S to be of the form

$$S = \left\{(x,y) \in \mathbb{Z}^2 \mid m \le x^2 + y^2 \le n\right\}$$

for some positive integers m and n, then Bob can win. (Bob does not know in advance that S is of this form.)