IMO 2017 Solution Notes

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This is an unofficial solutions packet for the 2017 IMO. In general, they are a combination of my own work, as well as the official solutions provided by the organizers (for which they hold any copyrights), and solutions found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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§0 Problems

1. For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \dots,$ by
$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise} \end{cases}$$
for each $n \ge 0$. Determine all values of a_0 for which there is a number A such that $a_n = A$ for infinitely many values of n.

2. Solve over $\mathbb{R}$ the functional equation
$$f\left(f(x)f(y)\right) + f(x+y) = f(xy).$$

3. A hunter and an invisible rabbit play a game in the plane. The rabbit and hunter start at points $A_0 = B_0$. In the nth round of the game ($n \ge 1$), three things occur in order:
   (i) The rabbit moves invisibly from A_{n-1} to a point A_n such that $A_{n-1}A_n = 1$.
   (ii) The hunter has a tracking device (e.g. dog) which reports an approximate location P_n of the rabbit, such that $P_nA_n \le 1$.
   (iii) The hunter moves visibly from B_{n-1} to a point B_n such that $B_{n-1}B_n = 1$.

   Let $N = 10^9$. Can the hunter guarantee that $A_NB_N < 100$?

4. Let R and S be different points on a circle Ω such that $\overline{RS}$ is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of $\overline{RT}$. Point J is chosen on minor arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ closer to R. Line AJ meets Ω again at K. Prove that line KT is tangent to Γ.

5. Fix $N \ge 1$. A collection of $N(N+1)$ soccer players of distinct heights stand in a row. Sir Alex Song wishes to remove $N(N-1)$ players from this row to obtain a new row of $2N$ players in which the following N conditions hold: no one stands behind the two tallest players, no one stands behind the third and fourth tallest players, ..., no one stands between the two shortest players. Prove that this is possible.

6. An *irreducible lattice point* is an ordered pair of integers (x, y) satisfying $\gcd(x, y) = 1$. Prove that if S is a finite set of irreducible lattice points then there exists a *homogeneous* polynomial $f(x, y)$ of degree at least 1 such that $f(x, y) = 1$ for each $(x, y) \in S$.

§1 IMO 2017/1, proposed by Stephan Wagner SAF

For each integer $a_0 > 1$, define the sequence a_0, a_1, a_2, …, by

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise} \end{cases}$$

for each $n \ge 0$. Determine all values of a_0 for which there is a number A such that $a_n = A$ for infinitely many values of n.

The answer is $a_0 \equiv 0 \pmod 3$ only.

First solution We first compute the minimal term of any sequence, periodic or not.

Lemma

Let c be the smallest term in a_n. Then either $c \equiv 2 \pmod 3$ or $c = 3$.

Proof. Clearly $c \neq 1, 4$. Assume $c \not\equiv 2 \pmod 3$ and hence $c \neq 2$. As c is not itself a square, the next perfect square after c in the sequence is one of $(\lfloor\sqrt{c}\rfloor + 1)^2$ $(\lfloor\sqrt{c}\rfloor + 2)^2$, or $(\lfloor\sqrt{c}\rfloor + 3)^2$. So by minimality we require

$$c \le \left\lfloor\sqrt{c}\right\rfloor + 3 < \sqrt{c} + 4$$

which requires $c \le 4$. Since $c \neq 1, 2, 4$ we conclude $c = 3$. ☐

Now we split the problem into two cases:

- If $a_0 \equiv 0 \pmod 3$, then all terms of the sequence are $0 \pmod 3$. The smallest term of the sequence is thus 3 by the lemma and we have

  $$3 \to 6 \to 9 \to 3$$

  so $A = 3$ works fine.
- If $a_0 \not\equiv 0 \pmod 3$, then no term of the sequence is $0 \pmod 3$, and so in particular 3 does not appear in the sequence. So the smallest term of the sequence is $2 \pmod 3$ by lemma. But since no squares are $2 \pmod 3$, the sequence a_k grows without bound forever after, so no such A can exist.

Hence the answer is $a_0 \equiv 0 \pmod 3$ only.

Second solution We clean up the argument by proving the following lemma.

Lemma

If a_n is constant modulo 3 and not $2 \pmod 3$, then a_n must eventually cycle in the form $(m, m+3, m+6, \dots, m^2)$, with no squares inside the cycle except m^2.

Proof. Observe that a_n must eventually hit a square, say $a_k = c^2$; the next term is $a_{k+1} = c$. Then it is forever impossible to exceed c^2 again, by what is essentially discrete intermediate value theorem. Indeed, suppose $a_\ell > c^2$ and take $\ell > k$ minimal (in particular $a_\ell \neq \sqrt{a_{\ell-1}}$). Thus $a_{\ell-1} \in \{c^2-2, c^2-1, c^2\}$ and thus for modulo 3 reasons we have $a_{\ell-1} = c^2$. But that should imply $a_\ell = c < c^2$, contradiction.

We therefore conclude $\sup\{a_n, a_{n+1}, \dots\}$ is a decreasing integer sequence in n. It must eventually stabilize, say at m^2. Now we can't hit a square between m and m^2, and so we are done. ☐

Now, we contend that all $a_0 \equiv 0 \pmod 3$ work. Indeed, for such a_0 we have $a_n \equiv 0 \pmod 3$ for all n, so the lemma implies that the problem statement is valid.

Next, we observe that if $a_i \equiv 2 \pmod 3$, then the sequence grows without bound afterwards since no squares are $2 \pmod 3$. In particular, if $a_0 \equiv 2 \pmod 3$ the answer is no.

Finally, we claim that if $a_0 \equiv 1 \pmod 3$, then eventually some term is $2 \pmod 3$. Assume for contradiction this is not so; then $a_n \equiv 1 \pmod 3$ must hold forever, and the lemma applies to give us a cycle of the form $(m, m+3, \dots, m^2)$ where $m \equiv 1 \pmod 3$. In particular $m \geq 4$ and

$$m \leq (m-2)^2 < m^2$$

but $(m-2)^2 \equiv 1 \pmod 3$ which is a contradiction.

§2 IMO 2017/2, proposed by Dorlir Ahmeti ALB

Solve over $\mathbb{R}$ the functional equation

$$f\left(f(x)f(y)\right) + f(x+y) = f(xy).$$

The only solutions are $f(x) = 0$, $f(x) = x - 1$ and $f(x) = 1 - x$, which clearly work. Note that

- If f is a solution, so is $-f$.
- Moreover, if $f(0) = 0$ then setting $y = 0$ gives $f \equiv 0$. So henceforth we assume $f(0) > 0$.

Lemma 2.1

$f(z) = 0 \iff z = 1$. Also, $f(0) = 1$ and $f(1) = 0$.

Proof. For the forwards direction, if $f(z) = 0$ and $z \neq 1$ one may put $(x, y) = \left(z, z(z-1)^{-1}\right)$ (so that $x + y = xy$) we deduce $f(0) = 0$ which is a contradiction.

For the reverse, $f(f(0)^2) = 0$ by setting $x = y = 0$, and use the previous part. We also conclude $f(1) = 0$, $f(0) = 1$. □

Lemma 2.2

If f is injective, we are done.

Proof. Setting $y = 0$ in the original equation gives $f(f(x)) = 1 - f(x)$. We apply this three times on the expression $f^3(x)$:

$$f(1 - f(x)) = f(f(f(x))) = 1 - f(f(x)) = f(x).$$

Hence $1 - f(x) = x$ or $f(x) = 1 - x$. □

Lemma 2.3

f is injective.

Proof. Now setting $y = 1$ in the original equation gives $f(x+1) = f(x) - 1$, and by induction

$$f(x+n) = f(x) - n. \qquad (1)$$

Assume now $f(a) = f(b)$. By using (1) we may shift a and b to be large enough that we may find x and y obeying $x + y = a + 1$, $xy = b$. Setting these gives

$$\begin{aligned} f(f(x)f(y)) &= f(xy) - f(x+y) = f(b) - f(a+1) \\ &= f(b) + 1 - f(a) = 1 \end{aligned}$$

so $f(x)f(y) = 0$ by the claim, hence $1 \in \{x, y\}$. But that implies $a = b$. □

> **Remark.** The result $f(f(x)) + f(x) = 1$ also implies that surjectivity would solve the problem.

> **Remark.** One can solve the problem over $\mathbb{Q}$ using only (1) and the easy parts. Indeed, that already implies $f(n) = 1 - n$ for all n. Now we induct to show $f(p/q) = 1 - p/q$ for all $0 < p < q$ (on q). By choosing $x = 1 + p/q$, $y = 1 + q/p$, we cause $xy = x + y$, and hence $0 = f\left(f(1 + p/q)f(1 + q/p)\right)$ or $1 = f(1 + p/q)f(1 + q/p)$.
>
> By induction we compute $f(1 + q/p)$ and this gives $f(p/q + 1) = f(p/q) - 1$.

§3 IMO 2017/3, proposed by Gerhard Woeginger AUT

A hunter and an invisible rabbit play a game in the plane. The rabbit and hunter start at points $A_0 = B_0$. In the nth round of the game ($n \ge 1$), three things occur in order:

(i) The rabbit moves invisibly from A_{n-1} to a point A_n such that $A_{n-1}A_n = 1$.

(ii) The hunter has a tracking device (e.g. dog) which reports an approximate location P_n of the rabbit, such that $P_nA_n \le 1$.

(iii) The hunter moves visibly from B_{n-1} to a point B_n such that $B_{n-1}B_n = 1$.

Let $N = 10^9$. Can the hunter guarantee that $A_NB_N < 100$?

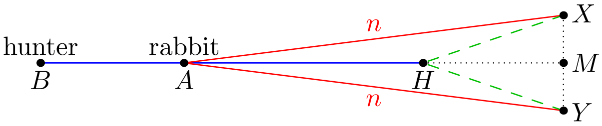
No, the hunter cannot. We will show how to increase the distance in the following way:

Claim — Suppose the rabbit is at a distance $d \ge 1$ from the hunter at some point in time. Then it can increase its distance to at least $\sqrt{d^2 + \frac{1}{2}}$ in $4d$ steps regardless of what the hunter already knows about the rabbit.

Proof. Consider a positive integer $n > d$, to be chosen later. Let the hunter start at B and the rabbit at A, as shown. Let ℓ denote line AB.

Now, we may assume the rabbit reveals its location A, so that all previous information becomes irrelevant.

The rabbit chooses two points X and Y symmetric about ℓ such that $XY = 2$ and $AX = AY = n$, as shown. The rabbit can then hop to either X or Y, pinging the point P_n on the ℓ each time. This takes n hops.



Now among all points H the hunter can go to, $\min\max\{HX, HY\}$ is clearly minimized with $H \in \ell$ by symmetry. So the hunter moves to a point H such that $BH = n$ as well. In that case the new distance is $HX = HY$.

We now compute

$$\begin{aligned} HX^2 &= 1 + HM^2 = 1 + \left(\sqrt{AX^2 - 1} - AH\right)^2 \\ &= 1 + \left(\sqrt{n^2 - 1} - (n - d)\right)^2 \\ &\ge 1 + \left(\left(n - \frac{1}{n}\right) - (n - d)\right)^2 \\ &= 1 + (d - 1/n)^2 \end{aligned}$$

which exceeds $d^2 + \frac{1}{2}$ whenever $n \ge 4d$. $\square$

In particular we can always take $n = 400$ even very crudely; applying the lemma $2 \cdot 100^2$ times, this gives a bound of $400 \cdot 2 \cdot 100^2 < 10^9$, as desired.

Remark. The step of revealing the location of the rabbit seems critical because as far as I am aware it is basically impossible to keep track of ping locations in the problem.

Remark. Reasons to believe the answer is "no": the 10^9 constant, and also that "follow the last ping" is losing for the hunter.

Remark. I think there are roughly two ways you can approach the problem once you recognize the answer.

(i) Try and control the location of the pings

(ii) Abandon the notion of controlling possible locations, and try to increase the distance by a little bit, say from d to $\sqrt{d^2 + \varepsilon}$. This involves revealing the location of the rabbit before each iteration of several jumps.

I think it's clear that the difficulty of my approach is realizing that (ii) is possible; once you do, the two-point approach is more or less the only one possible.

My opinion is that (ii) is not that magical; as I said it was the first idea I had. But I am biased, because when I test-solved the problem at the IMO it was called "C5" and not "IMO3"; this effectively told me it was unlikely that the official solution was along the lines of (i), because otherwise it would have been placed much later in the shortlist.

§4 IMO 2017/4, proposed by Charles Leytem LUX

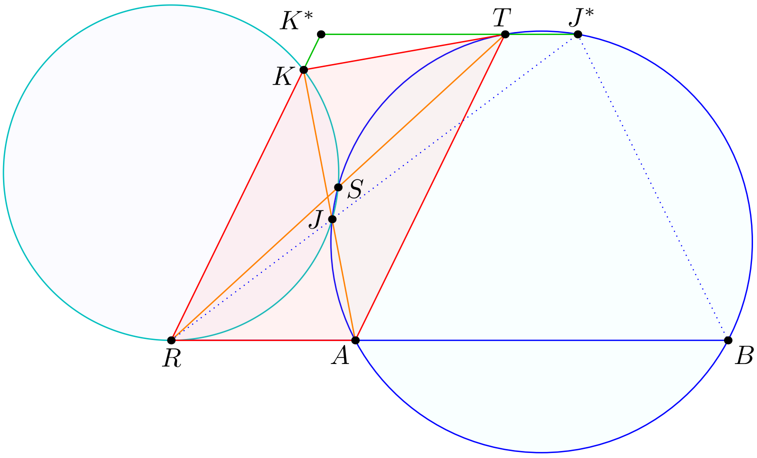
Let R and S be different points on a circle Ω such that $\overline{RS}$ is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of $\overline{RT}$. Point J is chosen on minor arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ closer to R. Line AJ meets Ω again at K. Prove that line KT is tangent to Γ.

First solution (elementary) First, note

$$\measuredangle RKA = \measuredangle RKJ = \measuredangle RSJ = \measuredangle TSJ = \measuredangle TAJ = \measuredangle TAK$$

so $\overline{RK} \parallel \overline{AT}$. Now,

- $\overline{RA}$ is tangent at R iff $\triangle KRS \sim \triangle RTA$ (oppositely), because both equate to $-\measuredangle RKS = \measuredangle SKR = \measuredangle SRA = \measuredangle TRA$.
- Similarly, $\overline{TK}$ is tangent at T iff $\triangle KTS \sim \triangle ART$.
- The two similarities are equivalent because $RS = ST$ the SAS gives $KR \cdot TA = RS \cdot RT = TS \cdot TR$.



> **Remark.** The problem is actually symmetric with respect to two circles; $\overline{RA}$ is tangent at R if and only if $\overline{TK}$ at T.

Second solution (inversion) Consider an inversion at R fixing the circumcircle Γ of $TSJA$. Then:

- T and S swap,
- A and B swap, where B is the second intersection of ℓ with Γ.
- Circle Ω inverts to the line through T parallel to $\overline{RAB}$, call it ℓ.
- J^* is the second intersection of ℓ with Γ.
- K^* is the intersection of ℓ with the circumcircle of RBJ^*; this implies RK^*J^*B is an isosceles trapezoid. In particular, one reads $\overline{RK^*} \parallel \overline{AT}$ from this, hence RK^*TA is a parallelogram.

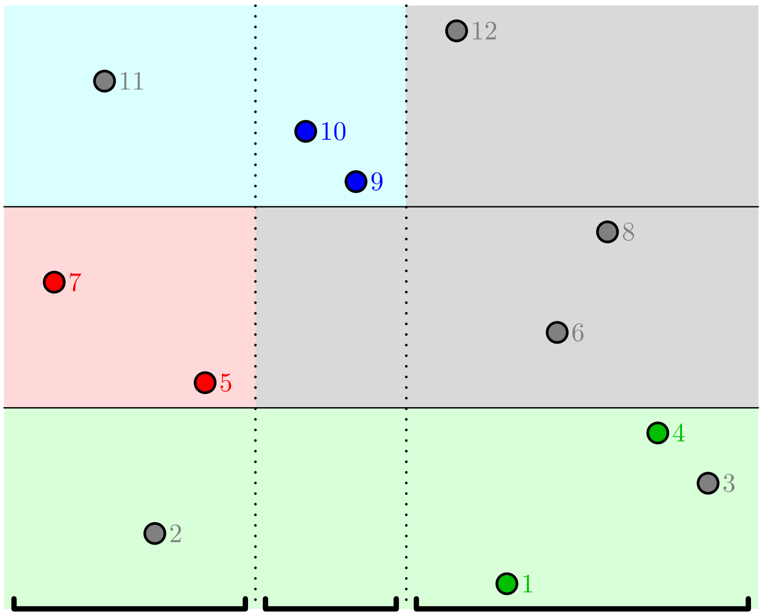
Thus we wish to show the circumcircle of RSK^* is tangent to Γ. But that follows from the final parallelogram observed: S is the center of the parallelogram since it is the midpoint of the diagonal.

> **Remark.** This also implies $RKTB$ is cyclic, from $\overline{K^*SA}$ collinear. Moreover, quadrilateral KK^*TS is cyclic (by power of a point); this leads to the second official solution to the problem.

§5 IMO 2017/5, proposed by Grigory Chelnokov RUS

Fix $N \ge 1$. A collection of $N(N+1)$ soccer players of distinct heights stand in a row. Sir Alex Song wishes to remove $N(N-1)$ players from this row to obtain a new row of $2N$ players in which the following N conditions hold: no one stands behind the two tallest players, no one stands behind the third and fourth tallest players, . . . , no one stands between the two shortest players. Prove that this is possible.

Some opening remarks: **location and height are symmetric to each other**, if one thinks about this problem as permutation pattern avoidance. So while officially there are multiple solutions, they are basically isomorphic to one another, and I am not aware of any solution otherwise.



Take a partition of N groups in order by height: $G_1 = \{1, \dots, N+1\}$, $G_2 = \{N+2, \dots, 2N+2\}$, and so on. We will pick two people from each group G_k.

Scan from the left until we find two people in the same group G_k. Delete all people scanned and also everyone in G_k. All the groups still have at least $N+1$ people left, so we can induct down with the non-deleted people; the chosen pair is to the far left anyways.

> **Remark.** The important bit is to *scan by position* but *group by height*, and moreover not change the groups as we scan. Dually, one can have a solution which scans by height but groups by position.

§6 IMO 2017/6, proposed by John Berman USA

An *irreducible lattice point* is an ordered pair of integers (x, y) satisfying $\gcd(x, y) = 1$. Prove that if S is a finite set of irreducible lattice points then there exists a *homogeneous* polynomial $f(x, y)$ of degree at least 1 such that $f(x, y) = 1$ for each $(x, y) \in S$.

We present two solutions.

First solution (Dan Carmon, Israel) We prove the result by induction on $|S|$, with the base case being Bezout's Lemma ($n = 1$). For the inductive step, suppose we want to add a given pair (a_{m+1}, b_{m+1}) to $\{(a_1, \dots, a_m), (b_1, \dots, b_m)\}$. By a suitable linear transformation assume $(a_{m+1}, b_{m+1}) = (1, 0)$. (The transformation is not necessary to proceed but cleans up the presentation that follows.)

Let $g(x, y)$ be a polynomial which works on the latter set. We claim we can choose the new polynomial f of the form

$$f(x, y) = g(x, y)^M - Cx^{\deg g \cdot M - m} \prod_{i=1}^{m} (b_i x - a_i y).$$

where C and M are integer parameters we may adjust.

Since $f(a_i, b_i) = 1$ by construction we just need

$$1 = f(1, 0) = g(1, 0)^M - C \prod b_i.$$

If $\prod b_i = 0$ we are done, since $b_i = 0 \implies a_i = \pm 1$ in that case and so $g(1, 0) = \pm 1$, thus take $M = 2$. So it suffices to prove:

> **Claim** — We have $\gcd(g(1, 0), b_i) = 1$ when $b_i \neq 0$.

Proof. Fix i. If $b_i = 0$ then $a_i = \pm 1$ and $g(\pm 1, 0) = \pm 1$. Otherwise know

$$1 = g(a_i, b_i) \equiv g(a_i, 0) \pmod{b_i}$$

and since the polynomial is homogeneous with $\gcd(a_i, b_i) = 1$ it follows $g(1, 0) \not\equiv 0 \pmod{b_i}$ as well. ☐

Then take M a large even multiple of $\varphi(\prod b_i)$ and we're done.

Second solution (Lagrange) The main claim is that:

> **Claim** — For every positive integer N, there is a homogeneous polynomial $P(x, y)$ such that $P(x, y) \equiv 1 \pmod{N}$ whenever $\gcd(x, y) = 1$.

(This claim is actually implied by the problem.)

Proof. For $N = p^e$ a prime take $(x^{p-1} + y^{p-1})^{\varphi(N)}$ when p is odd, and $(x^2 + xy + y^2)^{\varphi(N)}$ for $p = 2$.

Now suppose $N = q_1 q_2 \dots q_k$ where q_i are prime powers. Look at the polynomial Q_i described above for $i = 1, \dots, k$. Now

$$\frac{N}{q_i} Q_i(x, y) \equiv \frac{N}{q_i} \pmod{N}$$

for all x and y; so we can put together the polynomials $\frac{N}{q_i} Q_i$ by Bézout lemma. ☐

Let $S = \{(a_i, b_i) \mid i = 1, \dots, m\}$. We have the natural homogeneous "Lagrange polynomials" $L_k(x, y) = \prod_{i \neq k} (b_i x - a_i y)$. Now let $N = \prod_k L_k(x_k, y_k)$ and take P as above. Then we can take a large power of P, and for each i subtract an appropriate multiple of $L_i(x, y)$.