

# IMO 2008 Solution Notes

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This is an compilation of solutions for the 2008 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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## §0 Problems

1. Let  $H$  be the orthocenter of an acute-angled triangle  $ABC$ . The circle  $\Gamma_A$  centered at the midpoint of  $\overline{BC}$  and passing through  $H$  intersects the sideline  $BC$  at points  $A_1$  and  $A_2$ . Similarly, define the points  $B_1, B_2, C_1$ , and  $C_2$ . Prove that six points  $A_1, A_2, B_1, B_2, C_1, C_2$  are concyclic.

2. Let  $x, y, z$  be real numbers with  $xyz = 1$ , all different from 1. Prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$

and show that equality holds for infinitely many choices of rational numbers  $x, y, z$ .

3. Prove that there are infinitely many positive integers  $n$  such that  $n^2 + 1$  has a prime factor greater than  $2n + \sqrt{2n}$ .
4. Find all functions  $f$  from the positive reals to the positive reals such that

$$\frac{f(w)^2 + f(x)^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

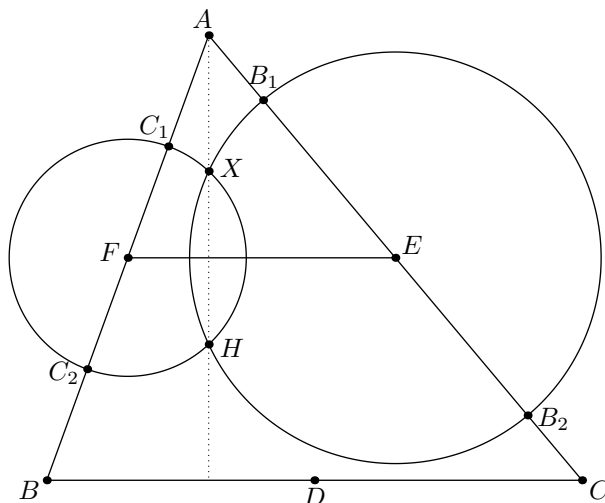
for all positive real numbers  $w, x, y, z$  satisfying  $wx = yz$ .

5. Let  $n$  and  $k$  be positive integers with  $k \geq n$  and  $k - n$  an even number. There are  $2n$  lamps labelled  $1, 2, \dots, 2n$  each of which can be either on or off. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on). Let  $N$  be the number of such sequences consisting of  $k$  steps and resulting in the state where lamps  $1$  through  $n$  are all on, and lamps  $n + 1$  through  $2n$  are all off. Let  $M$  be number of such sequences consisting of  $k$  steps, resulting in the state where lamps  $1$  through  $n$  are all on, and lamps  $n + 1$  through  $2n$  are all off, but where none of the lamps  $n + 1$  through  $2n$  is ever switched on. Determine  $\frac{N}{M}$ .
6. Let  $ABCD$  be a convex quadrilateral with  $BA \neq BC$ . Denote the incircles of triangles  $ABC$  and  $ADC$  by  $\omega_1$  and  $\omega_2$  respectively. Suppose that there exists a circle  $\omega$  tangent to ray  $BA$  beyond  $A$  and to the ray  $BC$  beyond  $C$ , which is also tangent to the lines  $AD$  and  $CD$ . Prove that the common external tangents to  $\omega_1$  and  $\omega_2$  intersect on  $\omega$ .

## §1 IMO 2008/1

Let  $H$  be the orthocenter of an acute-angled triangle  $ABC$ . The circle  $\Gamma_A$  centered at the midpoint of  $BC$  and passing through  $H$  intersects the sideline  $BC$  at points  $A_1$  and  $A_2$ . Similarly, define the points  $B_1, B_2, C_1,$  and  $C_2$ . Prove that six points  $A_1, A_2, B_1, B_2, C_1, C_2$  are concyclic.

Let  $D, E, F$  be the centers of  $\Gamma_A, \Gamma_B, \Gamma_C$  (in other words, the midpoints of the sides). We first show that  $B_1, B_2, C_1, C_2$  are concyclic. It suffices to prove that  $A$  lies on the radical axis of the circles  $\Gamma_B$  and  $\Gamma_C$ .



Let  $X$  be the second intersection of  $\Gamma_B$  and  $\Gamma_C$ . Clearly  $\overline{XH}$  is perpendicular to the line joining the centers of the circles, namely  $\overline{EF}$ . But  $\overline{EF} \parallel \overline{BC}$ , so  $\overline{XH} \perp \overline{BC}$ . Since  $\overline{AH} \perp \overline{BC}$  as well, we find that  $A, X, H$  are collinear, as needed.

Thus,  $B_1, B_2, C_1, C_2$  are concyclic. Similarly,  $C_1, C_2, A_1, A_2$  are concyclic, as are  $A_1, A_2, B_1, B_2$ . Now if any two of these three circles coincide, we are done; else the pairwise radical axii are not concurrent, contradiction. (Alternatively, one can argue directly that  $O$  is the center of all three circles, by taking the perpendicular bisectors.)

## §2 IMO 2008/2

Let  $x, y, z$  be real numbers with  $xyz = 1$ , all different from 1. Prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$

and show that equality holds for infinitely many choices of rational numbers  $x, y, z$ .

Let  $x = a/b, y = b/c, z = c/a$ , so we want to show

$$\left(\frac{a}{a-b}\right)^2 + \left(\frac{b}{b-c}\right)^2 + \left(\frac{c}{c-a}\right)^2 \geq 1.$$

A very boring computation shows this is equivalent to

$$\frac{(a^2b + b^2c + c^2a - 3abc)^2}{(a-b)^2(b-c)^2(c-a)^2} \geq 0$$

which proves the inequality (and it is unsurprising we are in such a situation, given that there is an infinite curve of rationals).

For equality, it suffices to show there are infinitely many integer solutions to

$$a^2b + b^2c + c^2a = 3abc \iff \frac{a}{c} + \frac{b}{a} + \frac{c}{b} = 3$$

or equivalently that there are infinitely many rational solutions to

$$u + v + \frac{1}{uv} = 3.$$

For any  $0 \neq u \in \mathbb{Q}$  the real solution for  $u$  is

$$v = \frac{-u + (u-1)\sqrt{1-4/u} + 3}{2}$$

and there are certainly infinitely many rational numbers  $u$  for which  $1 - 4/u$  is a rational square (say,  $u = \frac{-4}{q^2-1}$  for  $q \neq \pm 1$  a rational number).

### §3 IMO 2008/3

Prove that there are infinitely many positive integers  $n$  such that  $n^2 + 1$  has a prime factor greater than  $2n + \sqrt{2n}$ .

The idea is to pick the prime  $p$  first!

Select any large prime  $p \geq 2013$ , and let  $h = \lceil \sqrt{p} \rceil$ . We will try to find an  $n$  such that

$$n \leq \frac{1}{2}(p - h) \quad \text{and} \quad p \mid n^2 + 1.$$

This implies  $p \geq 2n + \sqrt{p}$  which is enough to ensure  $p \geq 2n + \sqrt{2n}$ .

Assume  $p \equiv 1 \pmod{8}$  henceforth. Then there exists some  $\frac{1}{2}p < x < p$  such that  $x^2 \equiv -1 \pmod{p}$ , and we set

$$x = \frac{p+1}{2} + t.$$

**Claim** — We have  $t \geq \frac{h-1}{2}$  and hence may take  $n = p - x$ .

*Proof.* Assume for contradiction this is false; then

$$\begin{aligned} 0 &\equiv 4(x^2 + 1) \pmod{p} \\ &= (p + 1 + 2t)^2 + 4 \\ &\equiv (2t + 1)^2 + 4 \pmod{p} \\ &< h^2 + 4 \end{aligned}$$

So we have that  $(2t + 1)^2 + 4$  is positive and divisible by  $p$ , yet at most  $\lceil \sqrt{p} \rceil^2 + 4 < 2p$ . So it must be the case that  $(2t + 1)^2 + 4 = p$ , but this has no solutions modulo 8.  $\square$

## §4 IMO 2008/4

Find all functions  $f$  from the positive reals to the positive reals such that

$$\frac{f(w)^2 + f(x)^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers  $w, x, y, z$  satisfying  $wx = yz$ .

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The answers are  $f(x) \equiv x$  and  $f(x) \equiv 1/x$ . These work, so we show they are the only ones.

First, setting  $(t, t, t, t)$  gives  $f(t^2) = f(t)^2$ . In particular,  $f(1) = 0$ . Next, setting  $(t, 1, \sqrt{t}, \sqrt{t})$  gives

$$\frac{f(t)^2 + 1}{2f(t)} = \frac{t^2 + 1}{2t}$$

which as a quadratic implies  $f(t) \in \{t, 1/t\}$ .

Now assume  $f(a) = a$  and  $f(b) = 1/b$ . Setting  $(\sqrt{a}, \sqrt{b}, 1, \sqrt{ab})$  gives

$$\frac{a + 1/b}{f(ab) + 1} = \frac{a + b}{ab + 1}.$$

One can check the two cases on  $f(ab)$  each imply  $a = 1$  and  $b = 1$  respectively. Hence the only answers are those claimed.

**§5 IMO 2008/5**

Let  $n$  and  $k$  be positive integers with  $k \geq n$  and  $k - n$  an even number. There are  $2n$  lamps labelled  $1, 2, \dots, 2n$  each of which can be either on or off. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on). Let  $N$  be the number of such sequences consisting of  $k$  steps and resulting in the state where lamps  $1$  through  $n$  are all on, and lamps  $n + 1$  through  $2n$  are all off. Let  $M$  be number of such sequences consisting of  $k$  steps, resulting in the state where lamps  $1$  through  $n$  are all on, and lamps  $n + 1$  through  $2n$  are all off, but where none of the lamps  $n + 1$  through  $2n$  is ever switched on. Determine  $\frac{N}{M}$ .

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Answer is  $2^{k-n}$ .

We construct a map from  $N$ -sequences to  $M$ -sequences as follows: just change every instance of  $n + 1$  to  $1$ ,  $n + 2$  to  $2$ , and so on.

Clearly this is well-defined and surjective. (Example: suppose  $k = 9$ ,  $n = 3$ , and we denote the lamps by  $A, B, C, X, Y, Z$ . Then  $AXXBYYBYC \mapsto AAABBBBBC$ .)

We claim that every  $M$ -sequence has exactly  $2^{n-k}$  pre-images. Indeed, suppose that there are  $c_1$  instances of lamp  $1$ . Then we want to pick an odd subset of the  $1$ 's to change to  $n + 1$ 's, so  $2^{c_1-1}$  ways.

(The hard part is finding the answer; the rest is a pretty clear bijection. Mainly looking at small cases is enough. Note how little certain things actually matter.)

### §6 IMO 2008/6

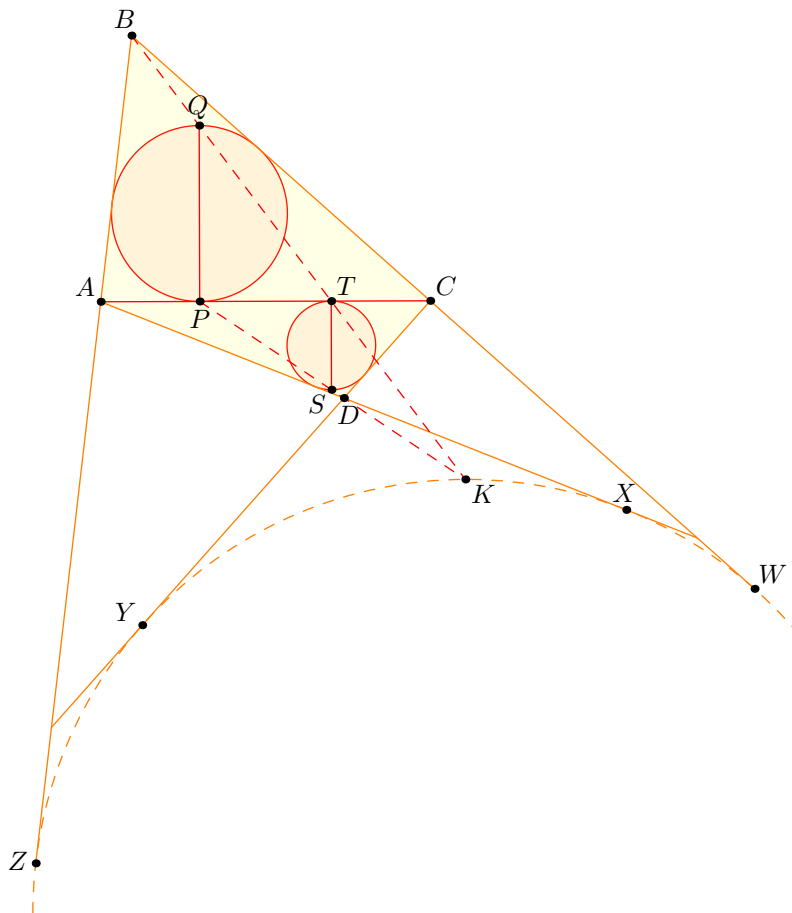
Let  $ABCD$  be a convex quadrilateral with  $BA \neq BC$ . Denote the incircles of triangles  $ABC$  and  $ADC$  by  $\omega_1$  and  $\omega_2$  respectively. Suppose that there exists a circle  $\omega$  tangent to ray  $BA$  beyond  $A$  and to the ray  $BC$  beyond  $C$ , which is also tangent to the lines  $AD$  and  $CD$ . Prove that the common external tangents to  $\omega_1$  and  $\omega_2$  intersect on  $\omega$ .

By the external version of Pitot theorem, the existence of  $\omega$  implies that

$$BA + AD = CB + CD.$$

Let  $\overline{PQ}$  and  $\overline{ST}$  be diameters of  $\omega_1$  and  $\omega_2$  with  $P, T \in \overline{AC}$ . Then the length relation on  $ABCD$  implies that  $P$  and  $T$  are reflections about the midpoint of  $\overline{AC}$ .

Now orient  $AC$  horizontally and let  $K$  be the “uppermost” point of  $\omega$ , as shown.



Consequently, a homothety at  $B$  maps  $Q, T, K$  to each other (since  $T$  is the uppermost of the excircle,  $Q$  of the incircle). Similarly, a homothety at  $D$  maps  $P, S, K$  to each other. As  $\overline{PQ}$  and  $\overline{ST}$  are parallel diameters it follows that  $K$  is the exsimilicenter of  $\omega_1$  and  $\omega_2$ .