Eyy LMaO

20th ELMO
Pittsburgh, PA

Year: **2018**

Day: **1**

Saturday, June 9, 2018
1:15PM — 5:45PM

Problem 1. Let n be a positive integer. There are $2018n + 1$ cities in the Kingdom of Sellke Arabia. King Mark wants to build two-way roads that connect certain pairs of cities such that for each city C and integer $1 \le i \le 2018$, there are exactly n cities that are a distance i away from C. (The *distance* between two cities is the least number of roads on any path between the two cities.)

For which n is it possible for Mark to achieve this?

Problem 2. Consider infinite sequences $a_1, a_2, \ldots$ of positive integers satisfying $a_1 = 1$ and

$$a_n \mid a_k + a_{k+1} + \cdots + a_{k+n-1}$$

for all positive integers k and n. For a given positive integer m, find the maximum possible value of a_{2m}.

Problem 3. Let A be a point in the plane, and ℓ a line not passing through A. Evan does not have a straightedge, but instead has a special compass which has the ability to draw a circle through three distinct noncollinear points. (The center of the circle is *not* marked in this process.) Additionally, Evan can mark the intersections between two objects drawn, and can mark an arbitrary point on a given object or on the plane.

(i) Can Evan construct* the reflection of A over ℓ?

(ii) Can Evan construct the foot of the altitude from A to ℓ?

*To construct a point, Evan must have an algorithm which marks the point in finitely many steps.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

Saturday, June 16, 2018
1:15PM — 5:45PM

Problem 4. Let ABC be a scalene triangle with orthocenter H and circumcenter O. Let P be the midpoint of $\overline{AH}$ and let T be on line BC with $\angle TAO = 90^\circ$. Let X be the foot of the altitude from O onto line PT. Prove that the midpoint of $\overline{PX}$ lies on the nine-point circle[†] of $\triangle ABC$.

Problem 5. Let $a_1, a_2, \ldots, a_m$ be a finite sequence of positive integers. Prove that there exist nonnegative integers b, c, and N such that

$$\left\lfloor \sum_{i=1}^{m} \sqrt{n+a_i} \right\rfloor = \left\lfloor \sqrt{bn+c} \right\rfloor$$

holds for all integers $n > N$.

Problem 6. A *windmill* is a closed line segment of unit length with a distinguished endpoint, the *pivot*. Let S be a finite set of n points such that the distance between any two points of S is greater than c. A configuration of n windmills is *admissible* if no two windmills intersect and each point of S is used exactly once as a pivot.

An admissible configuration of windmills is initially given to Geoff in the plane. In one operation Geoff can rotate any windmill around its pivot, either clockwise or counterclockwise and by any amount, as long as no two windmills intersect during the process. Show that Geoff can reach any other admissible configuration in finitely many operations, where

(i) $c = \sqrt{3}$,

(ii) $c = \sqrt{2}$.

[†]The nine-point circle of $\triangle ABC$ is the unique circle passing through the following nine points: the midpoint of the sides, the feet of the altitudes, and the midpoints of $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.