Shortlisted Problems

20^{th} ELMO

Pittsburgh, PA, 2018

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until disclosed publicly by the committee on the ELMO.

Contributing Students

The Problem Selection Committee for ELMO 2018 thanks the following proposers for contributing **90 problems** to this year's Competition:

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Problem Selection Committee

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Contents

Problems

A1. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a bijective function. Does there always exist an infinite number of functions $g \colon \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = g(f(x))$ for all $x \in \mathbb{R}$?

(Daniel Liu)

A2. Let $a_1, a_2, \dots, a_m$ be a finite sequence of positive integers. Prove that there exist nonnegative integers b, c, and N such that

$$\left\lfloor \sum_{i=1}^{m} \sqrt{n + a_i} \right\rfloor = \left\lfloor \sqrt{bn + c} \right\rfloor$$

holds for all integers $n > N$.

(Carl Schildkraut)

A3. Let a, b, c, x, y, z be positive reals such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$. Prove that

$$a^x + b^y + c^z \geq \frac{4abcxyz}{(x+y+z-3)^2}.$$

(Daniel Liu)

A4. Elmo calls a monic polynomial with real coefficients *tasty* if all of its coefficients are in $[-1, 1]$. A monic polynomial P with real coefficients and complex roots $\chi_1, \dots, \chi_m$ (counted with multiplicity) is given to Elmo, and he discovers that there does not exist a monic polynomial Q with real coefficients such that $P \cdot Q$ is tasty. Find all possible values of $\max(|\chi_1|, \dots, |\chi_m|)$.

(Carl Schildkraut)

C1. Let n be a positive integer. There are $2018n + 1$ cities in the Kingdom of Sellke Arabia. King Mark wants to build two-way roads that connect certain pairs of cities such that for each city C and integer $1 \leq i \leq 2018$, there are exactly n cities that are a distance i away from C. (The *distance* between two cities is the least number of roads on any path between the two cities.)

For which n is it possible for Mark to achieve this?

(Michael Ren)

C2. We say that a positive integer n is m-*expressible* if one can write a expression evaluating to n in base 10, where the expression consists only of

- exactly m numbers from the set $\{0, 1, \dots, 9\}$
- the six operations $+$, $-$, $\times$, $\div$, exponentiation ^, concatenation $\oplus$, and
- some number (possibly zero) of left and right parentheses.

For example, 5625 is 3-expressible (in two ways), as $5625 = 5 \oplus (5\string^4) = (7 \oplus 5)\string^2$, say. Does there exist a positive integer A such that all positive integers with A digits are $(A-1)$-expressible?

(Krit Boonsiriseth)

C3. A *windmill* in the plane consists of a line segment of unit length with a distinguished endpoint, the *pivot*. Geoff has a finite set of windmills, such that no two windmills intersect, and any two pivots are distance more than $\sqrt{2}$ apart. In an operation, Geoff can choose a windmill and rotate it about its pivot, either clockwise or counterclockwise and by any amount, as long as no two windmills intersect during or after the rotation. Show that Geoff can, in finitely many operations, rotate the windmills so that they all point in the same direction.

(Michael Ren)

G1. Let ABC be an acute triangle with orthocenter H, and let P be a point on the nine-point circle of ABC. Lines BH, CH meet the opposite sides AC, AB at E, F, respectively. Suppose that the circumcircles of $\triangle EHP$ and $\triangle FHP$ intersect lines CH, BH a second time at Q, R, respectively. Show that as P varies along the nine-point circle of ABC, the line QR passes through a fixed point.

(Brandon Wang)

G2. Let ABC be a scalene triangle with orthocenter H and circumcenter O. Let P be the midpoint of $\overline{AH}$ and let T be on line BC with $\angle TAO = 90^\circ$. Let X be the foot of the altitude from O onto line PT. Prove that the midpoint of $\overline{PX}$ lies on the nine-point circle of $\triangle ABC$.

(Zack Chroman)

G3. Let A be a point in the plane, and ℓ a line not passing through A. Evan doesn't have a straightedge, but instead has a special compass which has the ability to draw a circle through three distinct noncollinear points. (The center of the circle is *not* marked in this process.) Additionally, Evan can mark the intersections between two objects drawn, and can mark an arbitrary point on a given object or on the plane.

(i) Can Evan construct the reflection of A over ℓ?

(ii) Can Evan construct the foot of the altitude from A to ℓ?

(Zack Chroman)

G4. Let $ABCDEF$ be a convex hexagon inscribed in a circle Ω such that triangles ACE and BDF have the same orthocenter. Suppose that $\overline{BD}$ and $\overline{DF}$ intersect $\overline{CE}$ at X and Y, respectively. Show that there is a point common to Ω, the circumcircle of DXY, and the line through A perpendicular to $\overline{CE}$.

(Michael Ren and Vincent Huang)

G5. Let scalene triangle ABC have altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ and circumcenter O. The circumcircles of $\triangle ABC$ and $\triangle ADO$ meet at $P \neq A$. The circumcircle of $\triangle ABC$ meets lines $\overline{PE}$ at $X \neq P$ and $\overline{PF}$ at $Y \neq P$. Prove that $\overline{XY} \parallel \overline{BC}$.

(Daniel Hu)

N1. Determine all nonempty finite sets $S = \{a_1, \dots, a_n\}$ of n distinct positive integers such that $a_1 \cdots a_n$ divides $(x + a_1) \cdots (x + a_n)$ for every positive integer x.

(Ankan Bhattacharya)

N2. Call a number n *good* if it can be expressed in the form $2^x + y^2$ where x and y are nonnegative integers.

(a) Prove that there exist infinitely many sets of 4 consecutive good numbers.

(b) Find all sets of 5 consecutive good numbers.

(Michael Ma)

N3. Let $a_1, a_2, \dots$ be an infinite sequence of positive integers satisfying $a_1 = 1$ and

$$a_n \mid a_k + a_{k+1} + \cdots + a_{k+n-1}$$

for all positive integers k and n. Find the maximum possible value of a_{2018}.

(Krit Boonsiriseth)

N4. Fix a positive integer $n > 1$. We say a nonempty subset S of $\{0, 1, \dots, n-1\}$ is d-coverable if there exists a polynomial P with integer coefficients and degree at most d, such that S is exactly the set of residues modulo n that P attains as it ranges over the integers.

For each n, determine the smallest d such that any nonempty subset of $\{0, \dots, n-1\}$ is d-coverable, or prove that no such d exists.

(Carl Schildkraut)

Solutions

A1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bijective function. Does there always exist an infinite number of functions $g\colon \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = g(f(x))$ for all $x \in \mathbb{R}$?

(Daniel Liu)

Yes. It's clear $f^0, f^1, f^2, \dots$ all commute with f. If f doesn't have finite order this collection is infinite and valid.

Else, suppose that $f^n = \mathrm{id}$, where n is minimal. If $n = 1$ the problem is clear, so suppose $n > 1$. Then f is composed of some cycles; some cycle length $d \mid n$ appears infinitely many times. Let a countable number of these cycles be $x_{r,1} \to x_{r,2} \to \cdots \to x_{r,d} \to x_{r,1}$ for $r \in \mathbb{Z}$.

Then for every integer s, create a new function h_s fixing everything except the $x_{k,\ell}$, and send every $x_{r,a} \to x_{r+s,a}$. It is clear that every h_s commutes with f.

This gives infinitely many g, unless all but finitely many of the cycles have length 1. In that case, we can find more g by swapping any two fixed points of f and leaving everything else intact.

A2. Let $a_1, a_2, \dots, a_m$ be a finite sequence of positive integers. Prove that there exist nonnegative integers b, c, and N such that

$$\left\lfloor \sum_{i=1}^{m} \sqrt{n+a_i} \right\rfloor = \left\lfloor \sqrt{bn+c} \right\rfloor$$

holds for all integers $n > N$.

(Carl Schildkraut)

If all the a_i are equal, then $\sum_{i=1}^m \sqrt{n+a_i} = \sqrt{m^2 n + m^2 a_1}$ and so $(b,c) = (m^2, m^2 a_1)$ works fine.

Let us assume this is not the case. Instead, will take $b = m^2$ and $c = m(a_1 + \dots + a_m) - 1$ and claim it works for N large enough.

On the one hand,

$$\begin{aligned} \sum_{i=1}^{m} \sqrt{n+a_i} &< m \cdot \sqrt{n + \frac{a_1 + \dots + a_m}{m}} \\ &= \sqrt{m^2 \cdot n + c + 1} \le \left\lceil \sqrt{m^2 \cdot n + c + 1} \right\rceil \le \left\lfloor \sqrt{m^2 \cdot n + c} \right\rfloor + 1. \end{aligned}$$

On the other hand, let $\lambda = \frac{c}{2(c+1)} < \frac{1}{2}$. We use the following estimate.

Claim. If n is large enough in terms of $(a_1, \dots, a_n)$ then $\sqrt{n+a_i} \ge \sqrt{n} + \frac{\lambda a_i}{\sqrt{n}}$.

Proof. Squaring both sides, it's equivalent to $a_i \ge 2\lambda \cdot a_i + \frac{\lambda^2 a_i^2}{n}$, which holds for n big enough as $2\lambda < 1$. $\square$

Now,

$$\begin{aligned} \sum_{i=1}^{m} \sqrt{n+a_i} &\ge \sum_{i=1}^{m} \left(\sqrt{n} + \frac{\lambda a_i}{\sqrt{n}} \right) \\ &\ge m\sqrt{n} + \frac{\lambda \cdot (a_1 + \dots + a_n)}{\sqrt{n}} \\ &= m\sqrt{n} + \frac{\lambda \cdot (c+1)}{m\sqrt{n}} \\ &= m\sqrt{n} + \frac{c}{2m\sqrt{n}} > \sqrt{m^2 \cdot n + c} \qquad \ge \left\lfloor \sqrt{m^2 n + c} \right\rfloor. \end{aligned}$$

This finishes the problem.

Remark. Obviously, $b = m^2$ for asymptotic reasons (by taking n large). As for possible values of c:

- If $a_1 = \dots = a_m$, then one can show $c = m(a_1 + \dots + a_m)$ is the only valid choice. Indeed, taking n of the form $n = k^2 - a$ and $n = \frac{k^2-1}{m^2} - a$ is enough to see this.
- But if not all a_i are equal, the natural guess of taking $c = m(a_1 + \dots + a_n)$ is not valid in general. For example, we have that

  $$\left\lfloor \sqrt{n} + \sqrt{n+2} \right\rfloor \ne \left\lfloor \sqrt{4n+4} \right\rfloor \qquad n \in \{t^2 - 1 \mid t = 2, 3, \dots\}.$$

  I think one can actually figure out exactly which c are valid, though the answer will depend on some quadratic residues, and we do not pursue this line of thought here.

So any correct solutions must distinguish these two cases.

A3. Let a, b, c, x, y, z be positive reals such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$. Prove that

$$a^x+b^y+c^z \ge \frac{4abcxyz}{(x+y+z-3)^2}.$$

(Daniel Liu)

We present three solutions.

First solution, proof without words (by proposer)

$$\begin{aligned}
a^x+b^y+c^z &= yz\cdot\frac{a^x}{yz}+zx\cdot\frac{a^y}{zx}+xy\cdot\frac{a^z}{xy} \\
&\ge (xy+yz+zx)\left(\left(\frac{a^x}{yz}\right)^{yz}\left(\frac{b^y}{zx}\right)^{zx}\left(\frac{c^z}{xy}\right)^{xy}\right)^{\frac{1}{xy+yz+zx}} \\
&= (xy+yz+zx)\cdot\frac{(abc)^{\frac{xyz}{xy+yz+zx}}}{x^{\frac{xy+zx}{xy+yz+zx}}y^{\frac{yz+xy}{xy+yz+zx}}z^{\frac{zx+yz}{xy+yz+zx}}} \\
&\ge (xy+yz+zx)\cdot\frac{(abc)^{\frac{xyz}{xy+yz+zx}}}{\left(\frac{x\cdot\frac{xy+zx}{xy+yz+zx}+y\cdot\frac{yz+xy}{xy+yz+zx}+z\cdot\frac{zx+yz}{xy+yz+zx}}{2}\right)^2} \\
&= (xy+yz+zx)\cdot\frac{4(abc)^{\frac{xyz}{xy+yz+zx}}}{\left(\sum_{\text{cyc}}x\cdot\left(1-\frac{yz}{xy+yz+zx}\right)\right)^2} \\
&= \frac{4abc(xy+yz+zx)}{(x+y+z-3\frac{xyz}{xy+yz+zx})^2} \\
&= \frac{4abcxyz}{(x+y+z-3)^2}.
\end{aligned}$$

Second solution, by weighted AM-GM (Andrew Gu) By weighted AM-GM,

$$\frac{1}{x}\cdot xa^x+\frac{1}{y}\cdot yb^y+\frac{1}{z}\cdot zc^z \ge x^{\frac{1}{x}}y^{\frac{1}{y}}z^{\frac{1}{z}}abc.$$

Hence it suffices to show

$$x^{\frac{1}{x}}y^{\frac{1}{y}}z^{\frac{1}{z}} \ge \frac{4xyz}{(x+y+z-3)^2}.$$

By weighted AM-GM,

$$2x^{\frac{1}{2}(1-\frac{1}{x})}y^{\frac{1}{2}(1-\frac{1}{y})}z^{\frac{1}{2}(1-\frac{1}{z})} \le 2\cdot\frac{1}{2}(x-1+y-1+z-1)=x+y+z-3.$$

Squaring both sides and rearranging proves the required inequality.

Third solution, by Hölder and Schur/Muirhead (Evan Chen) By Hölder and weighted AM-GM we have

$$\sqrt{\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)(a^x+b^y+c^z)} \ge \frac{1}{x}\cdot a^{x/2}+\frac{1}{y}\cdot b^{y/2}+\frac{1}{z}\cdot c^{z/2} \ge (abc)^{1/2}.$$

Hence, it suffices to prove that

$$(x+y+z-3)^2 \ge 4xyz\left(1/x^2+1/y^2+1/z^2\right) \qquad \forall \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$$

which is a 3-variable symmetric inequality. It also happens to be is MOP 2011, K4.1, done in my SOS handout. We give a proof below (with $a=1/x$, etc).

Claim (Black MOP 2011, Test 4, Problem 1)**.** If $a,b,c>0$ then

$$\left((a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-3\right)^2 \ge 4\left(\frac{(a+b+c)(a^2+b^2+c^2)}{abc}\right)$$

Proof. Expanding and clearing denominators it's just

$$\sum_{\text{sym}} a^4b^2 + \sum_{\text{cyc}} a^3b^3 + 6a^2b^2c^2 \ge 2\sum_{\text{cyc}} a^4bc + 2\sum_{\text{sym}} a^3b^2c$$

which can also be written as

$$\begin{array}{ccccccccccccc}
 & & & & & & 0 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 1 & & -2 & & 1 & & & & \\
 & & & 2 & & -2 & & -2 & & 2 & & & \\
 & & 1 & & -2 & & 6 & & -2 & & 1 & & \\
 & 0 & & -2 & & -2 & & -2 & & -2 & & 0 & \\
0 & & 0 & & 1 & & 2 & & 1 & & 0 & & 0
\end{array}$$

in Chinese dumbass notation. This rewrites as

$$\sum_{\text{cyc}} a^4(b-c)^2 + 2\sum_{\text{cyc}} ab(ab-bc)(ab-ac) \ge 0$$

which is evident (the latter sum is "upsidedown triangle Schur"). □

A4. Elmo calls a monic polynomial with real coefficients *tasty* if all of its coefficients are in $[-1, 1]$. A monic polynomial P with real coefficients and complex roots $\chi_1, \dots, \chi_m$ (counted with multiplicity) is given to Elmo, and he discovers that there does not exist a monic polynomial Q with real coefficients such that $P \cdot Q$ is tasty. Find all possible values of $\max(|\chi_1|, \dots, |\chi_m|)$.

(Carl Schildkraut)

We claim the answer is $r > 1$.

Part 1: Any value of $r > 1$ can be achieved. To prove this, we will show that the polynomial

$$P(x) = x^n - r^n$$

has no tasty multiples if $r^n \geq 2$ (such an n exists because $r > 1$). Set $M = r^n$. Assume we have a polynomial

$$R(x) = \sum_{i=0}^{N} a_i x^i$$

so that $-1 \leq a_i \leq 1$ for all i ($a_N = 1$) and $P|R$. Taking R modulo P, we get that, with $N = bn + c$ and $0 \leq c < n$ (setting $a_k = 0$ if $k > N$),

$$R(x) = \sum_{j=0}^{n-1} \sum_{k=0}^{b} a_{kn+j} x^{kn+j} \equiv \sum_{j=0}^{n-1} x^j \left[\sum_{k=0}^{b} a_{kn+j} R^k \right].$$

We have this must be the zero polynomial (since $P|R$); specifically, taking $j = c$,

$$\sum_{k=0}^{b} a_{nk+c} R^k = 0$$

$$\sum_{k=0}^{b-1} (-a_{nk+c}) R^k = a_{bn+c} R^b$$

$$\sum_{k=0}^{b-1} |a_{nk+c}| R^k \geq R^b$$

(since $a_{bn+c} = a_N = 1$). However, since $|a_{nk+c}| \leq 1$, we then have

$$\sum_{k=0}^{b-1} R^k \geq R^b$$

$$\frac{R^b - 1}{R - 1} \geq R^b$$

$$R^b - 1 \geq R^{b+1} - R^b$$

$$R^b(2 - R) \geq 1.$$

However, as $R \geq 2$, this is false.

Part 2: Any polynomial with $r \le 1$ has a tasty multiple. Define the *sparsity* of a polynomial to be the greatest common divisor of the exponents m for which the coefficient of x^m in P is not zero. Equivalently, it is the largest integer d so that $P(x) = Q(x^d)$ for some polynomial Q.

We prove the following theorem:

Theorem. *Given any complex number z for which $|z| \le 1$, there exist tasty polynomials with z as a root that have arbitrarily large sparsities.*

Proof. Let $z = re^{i\theta}$. If θ is a rational multiple of π (say, $\theta = a\pi/b$), then we take the polynomial $x^{bn} - r^{bn}$ for any integer n; this has sparsity bn and is tasty (as $r \le 1$, $r^{bn} \le 1$). So, it suffices to prove this in the case where θ is not a rational multiple of π, and we henceforth assume this.

We claim that, for infinitely many n, the polynomial

$$x^{2n} - 2\cos(n\theta)\, r^n x^n + r^{2n}$$

is tasty (note that this polynomial has sparsity n and as such the theorem is implied by this claim). First note that this polynomial reduces to

$$x^n = r^n e^{\pm ni\theta} = \left(re^{\pm i\theta}\right)^n,$$

which is true at $x = re^{i\theta} = z$, so z is in fact a root.

We recall the following lemma:

Lemma. *For any real number ϕ which is not a rational multiple of π, the sequence $a_n = \cos(n\phi)$ has infinitely many terms in the range $[-1/2, 1/2]$.*

Indeed, let $\{x\}$ be the fractional part of x, and consider the sequence

$$\alpha_n = \left\{\frac{n\phi}{2\pi}\right\}.$$

We see that $-1/2 \le a_n \le 1/2$ iff $1/6 \le \alpha_n \le 1/3$ or $2/3 \le \alpha_n \le 5/6$. It is well known that the sequence $x_n = \{nx\}$ is dense in $[0, 1]$ for any irrational x, so this is true. Thus, for infinitely many n, as θ has been assumed not to be a rational multiple of π, the coefficients of P are bounded above in absolute value by r^n and r^{2n} for infinitely many n, both of which are ≤ 1 as $r \le 1$. □

We now provide a second lemma.

Lemma. *If $P(x)$ and $Q(x)$ are both tasty polynomials and the sparsity D of P is greater than the degree d of Q, then the product $R(x) = P(x)Q(x)$ is also tasty.*

Proof. Write

$$P(x) = \sum_{j=0}^{s} a_j x^{Dj}, \qquad Q(x) = \sum_{k=0}^{d} b_k x^k.$$

Then,

$$P(x)Q(x) = \sum_{j=0}^{s}\sum_{k=0}^{d} a_j b_k x^{Dj+k}.$$

As $D > d$, none of these terms interfere with one another (for each integer n, there is at most one choice of $0 \le j \le s,\ 0 \le k \le d$ so that $Dj + k = s$), so the coefficients of $R(x)$ are just the values of $a_j b_k$ as j and k range over the desired range; as each a_j and b_k are of magnitude ≤ 1, each pairwise product is as well, finishing the proof. □

Given a polynomial P with roots $\chi_1, \dots, \chi_m$ in $\mathbb{C}$ (possibly with duplicates), we will inductively construct the polynomial $R(x)$ that is tasty and that P divides. We define a sequence of polynomials $R_0, \dots, R_m$ so that $R_0(x) = 1$, and for each $0 < k \le m$, we take a tasty polynomial $P_k(x)$ with root χ_k and sparsity greater than the degree of R_{k-1}, and take $R_k(x) = R_{k-1}(x)P_k(x)$. Such a $P_k(x)$ is guaranteed to exist by our theorem, and the product $R_{k-1}(x)P_k(x)$ is guaranteed to be tasty by our lemma. Thus, we may take $R = R_m$, finishing the proof.

Remark. A polynomial P that has a tasty multiple exists for all $r < 2$: We have upon fixing $r < 2$ that for large enough n, we know $r^n - r^{n-1} - \cdots - r - 1 \le 0$. If n is minimal, $r^n - r^{n-1} - \cdots - r > 0$, and we can thus take some value $0 \le c \le 1$ for the constant term by the intermediate value theorem so that $P(x) = x^n - x^{n-1} - \cdots - x - c$ has a root at r. If $r \ge 2$, then $n = 1$ can be taken in Part 1 and thus no tasty multiples exist.

C1. Let n be a positive integer. There are $2018n + 1$ cities in the Kingdom of Sellke Arabia. King Mark wants to build two-way roads that connect certain pairs of cities such that for each city C and integer $1 \le i \le 2018$, there are exactly n cities that are a distance i away from C. (The *distance* between two cities is the least number of roads on any path between the two cities.)

For which n is it possible for Mark to achieve this?

(Michael Ren)

The answer is n even.

To see that n odd fails, note that by taking $i = 1$ we see the graph is n-regular; since it has an odd number of vertices we need n to be even.

On the other hand, if n is even, then consider the graph formed by taking the vertices of a regular $(2018n + 1)$-gon and drawing edges between vertices which are at most $n/2$ apart. Then this works.

C2. We say that a positive integer n is *m-expressible* if one can write a expression evaluating to n in base 10, where the expression consists only of

- exactly m numbers from the set $\{0, 1, \dots, 9\}$
- the six operations $+$, $-$, $\times$, $\div$, exponentiation ^, concatenation $\oplus$, and
- some number (possibly zero) of left and right parentheses.

For example, 5625 is 3-expressible (in two ways), as $5625 = 5 \oplus (5\hat{\ }4) = (7 \oplus 5)\hat{\ }2$, say. Does there exist a positive integer A such that all positive integers with A digits are $(A-1)$-expressible?

(Krit Boonsiriseth)

Here is a solution by Evan Chen achieving $A = 6 \cdot 10^6$, and reprising the joke "six consecutive zeros".

We will replace "exactly m numbers" with "at most m numbers", since this is the same. Suppose we group the digits of N into base 1000000, so that we have

$$N = s_1 s_2 s_3 \dots s_m$$

where each s_m is a group of six digits (s_1 padded with leading zeros, if needed, but $s_1 \neq \overline{000000}$). We consider two cases.

- Suppose some group is zero; then we find that N has six consecutive zeros in its decimal representations. Thus N has the form

$$N = X \oplus (b \cdot (1 \oplus 0)\hat{\ }6) \oplus Y$$

  for some strings X and Y (possibly empty), which are formed by repeated concatenation.

- Otherwise, note that $m \geq 10^6$. By a classical pigeonhole argument there exist indices $i < j$ such that $s_i + \dots + s_j \equiv 0 \pmod{999999}$. Let $n = \frac{1}{999999} s_i \dots s_j$. Then we can write

$$N = X \oplus \left[((1 \oplus 0)\hat{\ }6 - 1) \cdot n \right] \oplus Y$$

  for strings $X = s_1 \dots s_i$ and $Y = s_{j+1} \dots s_n$.

Remark (Possible motivational remarks)**.** Ankan Bhattacharya says: I knew that the answer had to be yes — the obvious counting argument to show answer no doesn't work, and the given elements are unrelated enough that proving a no answer would be very difficult.

Evan says: I think you really do have to use exponentiation, since otherwise the numbers aren't big enough; but exponentiation is really painful to deal with, so I tried to find a way to use it only once. This is less daunting than it seems because you can concatenate digits "for free" from a size perspective; thus you just need a substring that you can "save space" on. After a bit of guesswork I came upon the idea of taking modulo $10^6 - 1 = 999999$ (which saves about two digits) and from there I had it.

Remark (Possible motivational remarks)**.** Ankan Bhattacharya points out that if we fix all $N-2$ operations, then there are only 10^{N-1} choices, compared to $9 \cdot 10^{N-1}$ numbers we need to obtain. Thus we need to use different operations to reach different numbers. This suggests that all solutions are likely to use some amount of casework.

Unlike Ankan, I did not find the case split to be a substantial part of the problem. It came up naturally because I had an edge case where six consecutive zeros might appear in my argument, and the first case was patch-only in that situation.

C3. A *windmill* in the plane consists of a line segment of unit length with a distinguished endpoint, the *pivot*. Geoff has a finite set of windmills, such that no two windmills intersect, and any two pivots are distance more than $\sqrt{2}$ apart. In an operation, Geoff can choose a windmill and rotate it about its pivot, either clockwise or counterclockwise and by any amount, as long as no two windmills intersect during or after the rotation. Show that Geoff can, in finitely many operations, rotate the windmills so that they all point in the same direction.

(Michael Ren)

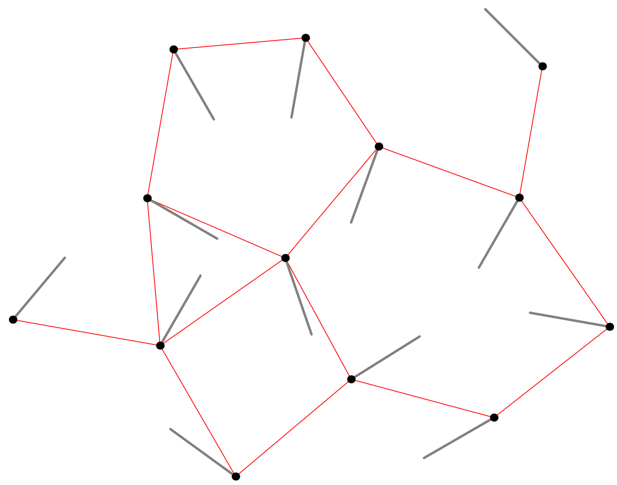
Throughout the solution we will general denote pivots by P, Q, R, ... and non-pivots by A, B, C, ...

We say that a configuration of windmills around S is *admissible* if no two windmills intersect. The problem is equivalent to showing one can reach any admissible configuration from any other (and the final position with the windmills pointing the same direction is just one example of a clearly admissible configuration).

Draw a red line segment between any two pivots which have distance at most 2 (thus these windmills could intersect). This naturally gives us a graph $\mathcal{G}$.

Lemma. *For $c \ge \sqrt{2}$, the graph $\mathcal{G}$ is planar.*

Proof. Indeed, if $\overline{PA}$ and $\overline{QB}$ intersect, we can consider convex quadrilateral $PQAB$, one of whose angles is at least 90°. WLOG it is $\angle PQA$, in which case $PA^2 \ge PQ^2 + QA^2 > 2 + 2 = 4$, so $\overline{PA}$ should not be red. $\square$



Clearly, we can ignore any isolated vertices. We can also ignore any leaves in $\mathcal{G}$; indeed suppose P is a pivot with $\overline{PQ}$ the only red edge. Then we can rotate the windmill at P to point away from Q and it will never obstruct other windmills since $c \ge 1$, so we can delete the pivot P from consideration (and use induction on the number of pivots, say).

Thus, we may assume $\mathcal{G}$ is a finite planar graph with no leaves. Thus it makes sense to speak of the faces of planar graph $\mathcal{G}$, consisting of several polygons.

Lemma. *A windmill with pivot P can never intersect a red edge other than those touching P.*

Proof. Suppose windmill $\overline{PA}$ intersects red edge $\overline{QR}$. Then the altitude from $\overline{PH}$ to $\overline{QR}$ has length at most 1. WLOG that $QH < RH$, so $QH < \frac{1}{2}QR = 1$. Then $PQ^2 < QH^2 + HP^2 < 1 + 1 = 2$, contradiction. $\square$

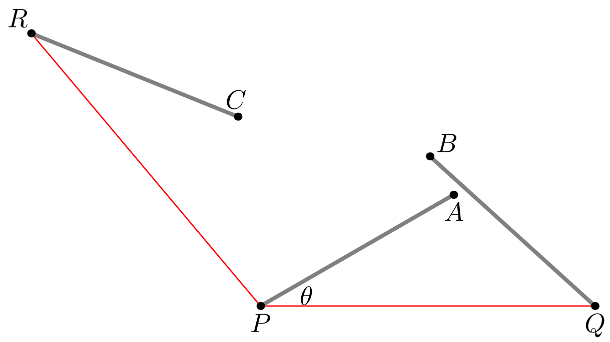
From now on, a windmill $\overline{PA}$ is said to *hug* a red edge $\overline{PQ}$ if the angle $\angle QPA < \varepsilon$ for some sufficiently small ε in terms of $\mathcal{G}$; each red edge $\overline{PQ}$ has at most two windmills hugging it (namely the windmills with pivots P and Q; if this happens, the windmills are on opposite sides of $\overline{PQ}$). Call a windmill configuration *cuddly* if every windmill is hugging an edge.

Claim. We can reach some cuddly configuration from any admissible one.

Proof. Indeed, consider a windmill $\overline{PA}$ not hugging any edge, and an edge $\overline{PQ}$, and such that $\angle APQ = \theta$ is minimal among all such pairs. Let $\angle RPQ$ be the corresponding angle of the face containing $\overline{PA}$, and let $\overline{QB}$, $\overline{RC}$ be windmills.

If $\overline{QB}$ is hugging $\overline{PQ}$, we perturb it slightly so that A and B are on opposite sides of $\overline{PQ}$; thus $\overline{QB}$ is no longer in the way.

We rotate $\overline{PA}$ towards $\overline{PQ}$ now. Because we assumed $\theta = \angle APQ$ was minimal, it is impossible for the body of the windmill to collide with the points B or C. So the only way it can be obstructed is if the point A collides with the interior of $\overline{QB}$ or $\overline{RC}$.

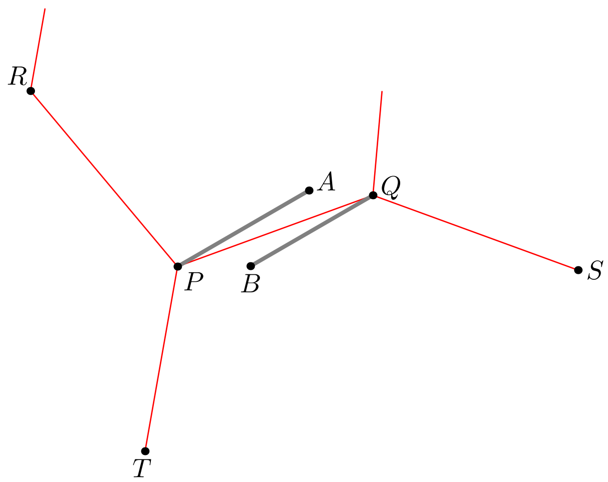


Suppose that A collided with $\overline{QB}$. At the moment of collision, we would have to have $\angle PAQ \le 90^\circ$. (This is because just before the collision $\overline{PA}$ was still disjoint from $\overline{QB}$, and if $\angle PAQ \ge 90^\circ$ just before then it would remain disjoint as $\overline{PA}$ rotated.) But then $PQ^2 \le PA^2 + AQ^2 \le 2$, contradiction. A similar proof works for $\overline{RC}$.

Thus we can rotate the windmills one by one so they hug the edges, as desired. $\square$

It remains to show any two cuddly configurations can be reached from each other. For this, we make two observations.

- Suppose $\overline{PA}$ and $\overline{QB}$ both hug $\overline{PQ}$. We show we can interchange the two. Assume $\angle RPQ$ is the angle of a face containing A, and $\angle TPQ$, $\angle PQS$ are the angles of the face containing B.



Rotate $\overline{PA}$ so it hugs $\overline{PR}$ (possibly perturbing the windmill at R), and then rotate $\overline{QB}$ so it hugs $\overline{QS}$ (possibly perturbing the windmill at S). Then rotate $\overline{PA}$ so it hugs $\overline{PT}$, then move $\overline{QB}$ back so it hugs $\overline{PQ}$ from the other side, and rotate $\overline{PA}$ back.

- Now suppose $\overline{PA}$ hugs $\overline{PQ}$, and $\angle RPQ$ is the angle of a face containing A. Then we can rotate it so that $\overline{PA}$ hugs $\overline{PR}$ (here $\overline{PA}$ could be blocked by $\overline{QB}$ initially, but then we perform the switching operation above).

Together these two observations finish the problem.

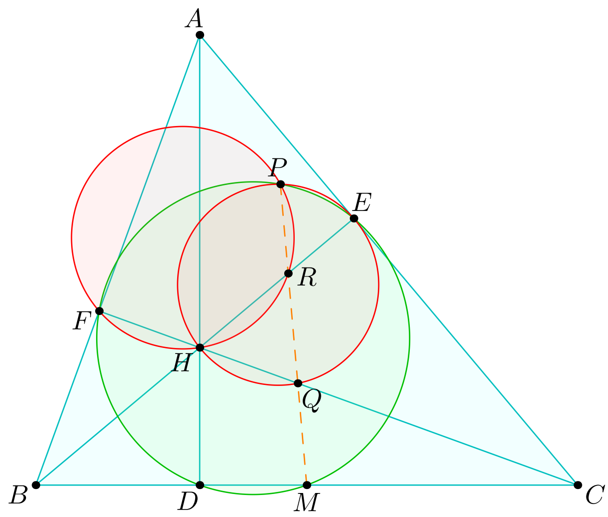
Remark (Michael Ren)**.** Here is a solution achieving just $c = \sqrt{3}$.

Draw a disk of radius $1 + \epsilon$ around every point in S such that the distance between any two points in S is more than $\sqrt{3}(1 + \epsilon)$ for some $\epsilon > 0$ that clearly exists. Note that no three disks can intersect. Indeed, if disks centered at A, B, and C intersected, then the circumradius of ABC is at most $1 + \epsilon$, which means that some two of A, B, C are at most a distance of $\sqrt{3}(1 + \epsilon)$ apart. In light of this, for any two points A and B in S that are a distance of at most 2 apart, draw a rhombus $APBQ$ of length $1 + \epsilon$. By our work before, all such rhombi are distinct. Furthermore, windmill collisions only happen inside these rhombi by definition. Now, have Geoff move each of his windmills one by one to Sasha's windmills. If a windmill collision happens, have Geoff move the other windmill out of the way inside the rhombus before moving the windmill by and then restore the position of the other windmill. Hence, he can always get his windmills to coincide, as desired.

G1. Let ABC be an acute triangle with orthocenter H, and let P be a point on the nine-point circle of ABC. Lines BH, CH meet the opposite sides AC, AB at E, F, respectively. Suppose that the circumcircles of $\triangle EHP$ and $\triangle FHP$ intersect lines CH, BH a second time at Q, R, respectively. Show that as P varies along the nine-point circle of ABC, the line QR passes through a fixed point.

(Brandon Wang)

Let D denote the foot of the A-altitude, and M the midpoint of $\overline{BC}$. We claim that R and Q both lie on line $\overline{PM}$. That will solve the problem (M is the fixed point).



By angle chasing, it is not hard to show that

$$\measuredangle FHE = \measuredangle FEM.$$

Now,

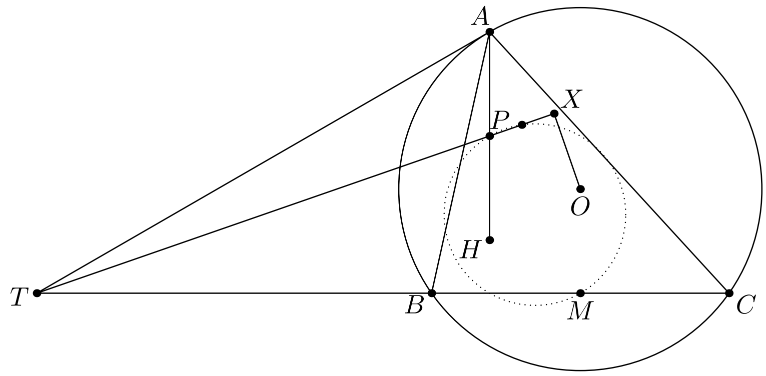
$$\measuredangle FPR = \measuredangle FHR = \measuredangle FHE = \measuredangle FEM = \measuredangle FPM$$

as desired so P, R, M are collinear. Similarly, P, Q, M are collinear, as desired.

G2. Let ABC be a scalene triangle with orthocenter H and circumcenter O. Let P be the midpoint of $\overline{AH}$ and let T be on line BC with $\angle TAO = 90^\circ$. Let X be the foot of the altitude from O onto line PT. Prove that the midpoint of $\overline{PX}$ lies on the nine-point circle of $\triangle ABC$.

(Zack Chroman)

We present two solutions, one synthetic and by complex numbers.



First solution (Zack Chroman) Let M be the midpoint of $\overline{BC}$. Note that since $\overline{AP} \perp \overline{BC}$ and $\overline{AT} \perp \overline{AO} \parallel \overline{PM}$, we find that P is the orthocenter of $\triangle ATM$. Thus $Y = \overline{TP} \cap \overline{AM}$ satisfies $\angle PYM = 90$, so it lies on the 9-point circle.

It then suffices to note that the reflection X' of P over Y lies on the circumcircle of $(AMT) = (TO)$, so $\angle TX'O = 90 \implies X = X'$.

Second solution (complex numbers, Evan Chen) Let Q denote the reflection of P over M, the midpoint of $\overline{BC}$.

Claim. We have $\overline{QO} \perp \overline{PT}$.

Proof. By complex numbers. We have

$$\begin{aligned} t &= \frac{aa(b+c) - bc(a+a)}{aa - bc} = \frac{a^2(b+c) - 2abc}{a^2 - bc} \\ t - p &= \frac{a^2(b+c) - 2abc}{a^2 - bc} - \left(a + \frac{b+c}{2}\right) \\ &= \frac{a^2(\frac{1}{2}b + \frac{1}{2}c - a) + (-a + \frac{1}{2}b + \frac{1}{2}c)bc}{a^2 - bc} \\ &= \frac{b+c-2a}{2} \cdot \frac{a^2+bc}{a^2-bc} \\ q &= 2 \cdot \frac{b+c}{2} - p = \frac{b+c-2a}{2} \end{aligned}$$

Since $\frac{a^2+bc}{a^2-bc} \in i\mathbb{R}$, the claim is proven. □

Thus, $\overline{QOX}$ are collinear. By considering right triangle $\triangle PQX$ with midpoint M, we conclude that $MX = MP$. Since the nine-point circle is the circle with diameter $\overline{PM}$, it passes through the midpoint of $\overline{PX}$.

G3. Let A be a point in the plane, and ℓ a line not passing through A. Evan doesn't have a straightedge, but instead has a special compass which has the ability to draw a circle through three distinct noncollinear points. (The center of the circle is *not* marked in this process.) Additionally, Evan can mark the intersections between two objects drawn, and can mark an arbitrary point on a given object or on the plane.

(i) Can Evan construct the reflection of A over ℓ?

(ii) Can Evan construct the foot of the altitude from A to ℓ?

(Zack Chroman)

The trick is to invert the figure around a circle centered at A of arbitrary radius. We let $\omega = \ell^*$ denote the image of ℓ under this inversion. Then, under the inversion, Evan's compass has the following behavior:

- Evan can draw a line through two points other than A; or
- Evan can draw a circle through three points other than A.

In other words, the point A is "invisible" to Evan, but Evan otherwise has a straightedge and the same compass.

It is clear then that the answer to (ii) is no.

Part (i) is equivalent to showing that Evan can construct the center of ω; we give one construction here anyways. Take any cyclic quadrilateral $WXYZ$ inscribed in ω, and let $P = \overline{WZ} \cap \overline{XY}$. Then the circumcircles of $\triangle PWX$ and $\triangle PYZ$ meet again at the Miquel point M, and the second intersection of (MXZ) and (MWY) is the center of ω.

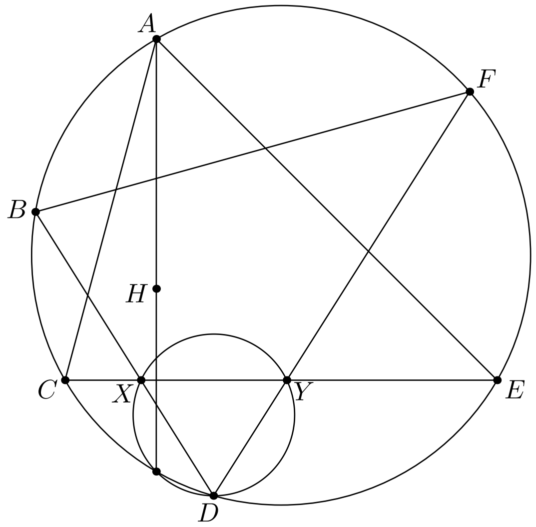
Remark. The proof of (ii) implies that it's actually more or less impossible in this context to construct any point other than the reflection of A, as a function of A and ℓ.

An alternative proof of (ii) is possible by inverting around a generic point P on ℓ with radius PA; this necessarily preserves the entire construction, but the foot from A to ℓ is not fixed by this inversion.

G4. Let $ABCDEF$ be a convex hexagon inscribed in a circle Ω such that triangles ACE and BDF have the same orthocenter. Suppose that $\overline{BD}$ and $\overline{DF}$ intersect $\overline{CE}$ at X and Y, respectively. Show that there is a point common to Ω, the circumcircle of DXY, and the line through A perpendicular to $\overline{CE}$.

(Michael Ren and Vincent Huang)

We present many, many solutions. In all of them, we let H denote the common orthocenter.



First solution by Simson lines (Vincent Huang) Let AH meet CE and Ω again at M and A_1, respectively, and P and Q be the projections of A_1 onto BD and DF, respectively. Note that PQ is the Simson line of A_1 with respect to BDF. It is well known that this Simson line bisects the segment between A_1 and H. Hence, M lies on PQ. But P, M, and Q are respectively the projections of A_1 onto DX, XY, and YD, so A_1 must lie on the circumcircle of DXY, as desired.

Second solution by dual Desargue involution (Michael Ren) Let O and r be the center and radius of Ω, respectively. Let $\mathcal{E}$ be the ellipse with foci O and H consisting of the set of points P such that $OP + HP = r$. Note that as the reflections of H over AC, CE, EA, BD, DF, FB lie on Ω, $\mathcal{E}$ is tangent to the sides of ACE and BDF. Let $\mathcal{E}$ and AD meet CE at P and Q, respectively. By the dual of Desargue's involution theorem on quadrilateral $ACPE$ with inscribed conic $\mathcal{E}$, $D(CE; XY; PQ)$ is an involution. Hence, the circumcircles of DCE, DXY, and DPQ are coaxial, so it suffices to show that A_1DPQ is cyclic, where A_1 is the second intersection of AH and Ω. But note that A_1 lies on OP, so $\angle QDA_1 = \angle ADA_1 = \frac{\pi}{2} - \angle OA_1A = \frac{\pi}{2} - \angle PA_1A$, which is the angle between PA_1 and PQ by the perpendicularity of AA_1 and CE, as desired.

Third solution by angle chasing (Mihir Singhal) Let A_1 be the reflection of H over CE. Note A_1 is on Ω so it suffices to show that DA_1XY is cyclic. Let M be the foot of the altitude from A to $\overline{CE}$. Note that M is the midpoint of $\overline{HA_1}$ so since A_1 is on Ω, M must be on the nine-point circle of DBF. Let R and S be the feet of the altitudes from F and B in DBF.

Note $MXRH$ and $MYSH$ are cyclic. Moreover, M lies on the nine-point circle of $\triangle BDF$, and hence $\measuredangle SMR = 2\measuredangle SDR$. Then

$$\begin{aligned}\measuredangle XHY &= \measuredangle XHM + \measuredangle MHY \\ &= \measuredangle XRM + \measuredangle MSY = \measuredangle DRM + \measuredangle MSD \\ &= -(\measuredangle RMS + \measuredangle SDR) = \measuredangle SMR + \measuredangle RDS \\ &= 2\measuredangle SDR + \measuredangle RDS = \measuredangle SDR = \measuredangle YDX.\end{aligned}$$

Thus $\measuredangle XA_1Y = -\measuredangle XHY = \measuredangle RDS = \measuredangle XDY$, as needed.

Fourth solution by inversion (James Lin) Let K be the second intersection of Ω and the perpendicular from A to CE. We want to show $DKXY$ is cyclic. We invert about H. It's clear that now, $A'C'E'$ and $B'D'F'$ share the same circumcircle Ω' and incenter H. Note that K maps to the midpoint $M_{A'}$ of the arc $C'E'$ on Ω' not containing A'. Also note that X' is the intersection of circles $(HB'D')$ and $(HC'E')$, which are centered at midpoint $M_{F'}$ of the arc $B'D'$ on Ω' not containing F' and the midpoint $M_{D'}$ of the arc $B'F'$ on Ω' not containing D', respectively. Thus, X' is the reflection of H over $M_{A'}M_{F'}$. Similarly, Y' is the reflection of H over $M_{A'}M_{B'}$. Then, note that $M_{A'}X = M_{A'}H = M_{A'}Y$. Now we reformulate the problem by erasing A', C' and E', as the rest of the problem can be defined without them. The reformulated statement is that if we fix B, D, F, H and vary $M_{A'}$ along Ω', then $D'M_{A'}X'Y'$ is always cyclic.

We proceed with directed angles. Note that $\measuredangle X'D'M_{A'} = \measuredangle X'D'H + \measuredangle HD'M_{A'} = \measuredangle M_{A'}M_{F'}F + \measuredangle M_{D'}M_{F'}M_{A'} = \measuredangle M_{D'}M_{F'}F$. Similarly, $\measuredangle Y'D'M_{A'} = M_{D'}M_{B'}B = -\measuredangle M_{D'}M_{F'}F = -\measuredangle X'DM_{A'}$, so it follows that $M_{A'}$ lies on an angle bisector of $\measuredangle X'D'Y'$. Assume that $D'M_{A'}$ and $X'Y'$ are not perpendicular. Then from $M_{A'}X' = M_{A'}Y'$, it follows that $D'M_{A'}X'$ and $D'M_{A'}Y'$ have the same circumradius, and if they don't have the same circumcircle, then $D'M_{A'}$ and $X'Y'$ must be perpendicular, a contradiction. So $D'X'M_{A'}Y'$ is cyclic. Hf $D'M_{A'}$ and $X'Y'$ are perpendicular, then use the new problem formulation (without A, C and E and just varying $M_{A'}$) to move $M_{A'}$ by a miniscule amount. Then $D'M_{A'}$ and $X'Y'$ will not be perpendicular, so $D'X'M_{A'}Y'$ is cyclic both after and before moving $M_{A'}$ by continuity. We are done.

Fifth solution, by complex numbers (Carl Schildkraut) Let Ω be the unit circle, and let $A = a$, etc. We have that

$$c + e = h - a \implies \frac{c+e}{ce} = \bar{h} - \frac{1}{a} \implies ce = \frac{a(h-a)}{a\bar{h} - 1}.$$

Let T be the second intersection of the line through A perpendicular to CE and Ω. We see that

$$t = -\frac{ce}{a} = -\frac{h-a}{a\bar{h}-1}.$$

We endeavor to show that $DTXY$ is a cyclic quadrilateral. We have that

$$\begin{aligned}
x &= \frac{ce(b+d)-bd(c+e)}{ce-bd} \\
&= \frac{\frac{a(b+d)(h-a)}{a\bar{h}-1} - bd(h-a)}{\frac{a(h-a)}{a\bar{h}-1} - bd} \\
&= (h-a)\left(\frac{a(b+d)-bd(a\bar{h}-1)}{a(h-a)-bd(a\bar{h}-1)}\right) \\
&= (h-a)\left(\frac{ab+ad-ab-ad-\frac{abd}{f}+bd}{ab+ad+af-a^2-ab-ad-\frac{abd}{f}+bd}\right) \\
&= (h-a)\left(\frac{bd(f-a)}{(af+bd)(f-a)}\right) \\
&= \frac{bd(h-a)}{af+bd}.
\end{aligned}$$

Similarly

$$y = \frac{bf(h-a)}{ab+df}.$$

So, we want to show that

$$d, -\frac{h-a}{a\bar{h}-1}, \frac{bd(h-a)}{af+bd}, \frac{bf(h-a)}{ab+df}$$

are concyclic. This is equivalent to, dividing each by $h-a$ and reciprocating,

$$\frac{h-a}{d}, 1-a\bar{h}, 1+\frac{af}{bd}, 1+\frac{ab}{df}$$

being concyclic. This is equivalent to, subtracting 1 and multiplying by bdf,

$$f(b+f-a), -a(bd+bf+df), ab^2, af^2$$

being concyclic. This is equivalent to, adding abf and dividing by $b+f$,

$$f, -ad, ab, af$$

being concyclic. However, all of these points lie on the unit circle, finishing the proof.

Sixth solution by complex numbers (Evan Chen) As usual let Ω denote the unit circle. We immediately have

$$\begin{aligned}
c+e &= b+d+f-a \\
\text{and thus} \quad \frac{1}{c}+\frac{1}{e} = \frac{c+e}{ce} &= \frac{1}{b}+\frac{1}{d}+\frac{1}{f}-\frac{1}{a} \\
\implies ce &= \frac{b+f+d-a}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}}.
\end{aligned}$$

These two equations let us eliminate c and e, leaving only a, b, d, f.

Now consider the point $p = -\frac{ce}{a}$ on the circumcircle. We compute

$$\begin{aligned}\frac{x-p}{b-p} &= \frac{x+\frac{ce}{a}}{b+\frac{ce}{a}} \\ &= \frac{\frac{bd(c+e)-ce(b+d)}{bd-ce}+\frac{ce}{a}}{b+\frac{ce}{a}} \\ &= \frac{abcd+abde-abce-adce+bdce-(ce)^2}{(ab+ce)(bd-ce)} \\ &= \frac{abcde(1/a+1/e+1/c-1/d-1/b)-(ce)^2}{(ab+ce)(bd-ce)} \\ &= \frac{abcde(1/f)-(ce)^2}{(ab+ce)(bd-ce)} = \frac{(ce)(abd-cef)}{f(ab+ce)(bd-ce)}\end{aligned}$$

Now, we write

$$\begin{aligned}ab+ce &= \frac{ab(\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a})+(b+f+d-a)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\ &= \frac{ab(\frac{1}{d}+\frac{1}{f})+d+f}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} = \frac{\frac{1}{df}(d+f)(ab+df)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\ bd-ce &= \frac{bd(\frac{1}{b}+\frac{1}{d}+\frac{1}{f}-\frac{1}{a})-(b+f+d-a)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\ &= \frac{bd(\frac{1}{f}-\frac{1}{a})+(a-f)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} = \frac{\frac{1}{af}(a-f)(bd+af)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\ abd-cef &= abd-\frac{f(b+f+d-a)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\ &= \frac{abd(\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a})-f(b+f+d-a)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\ &= \frac{(b+f)(\frac{abd}{bf}-f)+b(a-d)+f(a-d)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\ &= \frac{(b+f)(\frac{ad}{f}-f+(a-d))}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\ &= \frac{\frac{1}{f}(b+f)(a-f)(f+d)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}}.\end{aligned}$$

Putting that all together gives

$$\frac{x-p}{b-p} = \frac{ce\cdot adf(b+f)(\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a})}{(ab+df)(bd+af)}$$

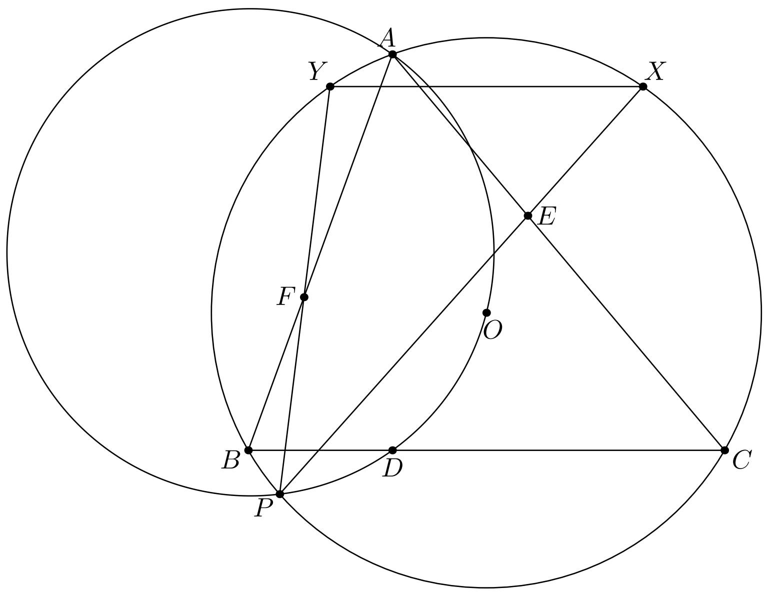
which is symmetric in d and f, so the analogous calculation with $\frac{y-p}{f-p}$ yields the same result. Consequently, P is the center of the spiral similarity sending $\overline{YF}$ to $\overline{BX}$, as desired.

Remark. Philosophical point: it's necessary to use both $a+c+e=b+d+f$ and its conjugate, to capture two degrees of freedom.

G5. Let scalene triangle ABC have altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ and circumcenter O. The circumcircles of $\triangle ABC$ and $\triangle ADO$ meet at $P \neq A$. The circumcircle of $\triangle ABC$ meets lines $\overline{PE}$ at $X \neq P$ and $\overline{PF}$ at $Y \neq P$. Prove that $\overline{XY} \parallel \overline{BC}$.

(Daniel Hu)

Denote by Ω and H the circumcircle and orthocenter of $\triangle ABC$. Let T lie on Ω such that $\overline{AT} \parallel \overline{BC}$. Let $\triangle ABC$ have orthocenter H.



First solution, synthetic First we prove a lemma.

Claim. The points H, P, T are collinear.

Proof. Let $\overline{HT}$ meet Ω at $P^* \neq T$. Let $\overline{AD}$ meet Ω at $K \neq A$. By homothety at K, $\overline{HT} \parallel \overline{DO}$. By angle chasing, $\angle P^*AD = \angle P^*AK = \angle P^*TK = \angle P^*TO = \angle OP^*T = \angle P^*OD$, so P^* lies on the circumcircle of $\triangle AOD$. Therefore, $P \equiv P^*$ as desired. $\square$

We now provide two finishes.

- First finish: By DDIT on $AEHF$, the pairs of lines $(\overline{PA}, \overline{PH}), (\overline{PB}, \overline{PC}), (\overline{PE}, \overline{PF})$ are part of a single involution, so $\overline{AT}, \overline{BC}, \overline{XY}$ are concurrent. Since $\overline{AT} \parallel \overline{BC}$, this implies that $\overline{XY} \parallel \overline{BC}$ as desired.

- Second finish: Let $Q = \overline{AP} \cap \overline{EF}$. By inversion at A, $BFPQ$, $CEPQ$, $DHPQ$ are all cyclic. By the lemma, this implies that $\angle ABC + \angle ACB = \angle APT = \angle APH = \angle QPH = \angle QDH = \angle QAH$, so $\overline{DQ} \perp \overline{EF}$.

  Let $G = \overline{EF} \cap \overline{BC}$; since $(G, D; B, C) = -1$, $\angle BQD = \angle DQC$. Thus $\angle BAY = \angle BPY = \angle BPF = \angle BQF = \angle CQE = \angle CPE = \angle CPX = \angle CAX$, so $\overline{XY} \parallel \overline{BC}$ as desired.

Second solution by complex numbers (Adam Ardeishar) Let ABC be the complex unit circle. Then $D = \frac{1}{2}(a+b+c-\frac{bc}{a})$, and we know

$$\frac{p-a}{p-o} \cdot \frac{d-o}{d-a} \in \mathbb{R}$$

$$\frac{p-a}{p} \cdot \frac{a+b+c-\frac{bc}{a}}{b+c-a-\frac{bc}{a}} = \frac{\frac{1}{p}-\frac{1}{a}}{\frac{1}{p}} \cdot \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{a}{bc}}{-\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{a}{bc}}$$

$$\frac{1}{p} \cdot \frac{a+b+c-\frac{bc}{a}}{b+c-a-\frac{bc}{a}} = \frac{-1}{a} \cdot \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{a}{bc}}{\frac{1}{b}+\frac{1}{c}-\frac{1}{a}-\frac{a}{bc}}$$

$$\frac{-a}{p} \cdot \frac{a^2+ab+ac-bc}{ab+ac-a^2-bc} = \frac{bc+ab+ab-a^2}{ab+ac-bc-a^2}$$

$$p = a \cdot \frac{a^2+ab+ac-bc}{a^2-ab-ac-bc}$$

Now note that $p + x = e + px\bar{e}$, so $x = \frac{p-e}{p\bar{e}-1}$ But we compute that

$$p - e = a \cdot \frac{a^2+ab+ac-bc}{a^2-ab-ac-bc} - \frac{1}{2}(a+b+c-\frac{ac}{b})$$

$$= \frac{a^3b + a^3 + 2a^2b^2 + a^2bc + ab^3 + ab^2c + b^3c + b^2c^2 - a^2c^2}{2b(a^2-ab-ac-bc)}$$

$$= \frac{(a+b)(b+c)(a^2+ab-ac+bc)}{2b(a^2-ab-ac-bc)}$$

And also compute

$$p\bar{e} - 1 = a \cdot \frac{a^2+ab+ac-bc}{a^2-ab-ac-bc} \cdot \frac{1}{2}(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{b}{ac}) - 1$$

$$= \frac{a^3b + a^3c + a^2bc + a^2c^2 + ab^2c + 2abc^2 + b^3c + b^2c^2 - ab^3}{2bc(a^2-ab-ac-bc)}$$

$$= \frac{(a+b)(b+c)(a^2+ac+bc-ab)}{2bc(a^2-ab-ac-bc)}$$

So

$$x = \frac{\frac{(a+b)(b+c)(a^2+ab-ac+bc)}{2b(a^2-ab-ac-bc)}}{\frac{(a+b)(b+c)(a^2+ac+bc-ab)}{2bc(a^2-ab-ac-bc)}} = c \cdot \frac{a^2+ab+bc-ac}{a^2+ac+bc-ab}$$

By symmetry,

$$y = b \cdot \frac{a^2+ac+bc-ab}{a^2+ab+bc-ac}$$

Now note that $xy = bc$ to finish.

N1. Determine all nonempty finite sets $S = \{a_1, \dots, a_n\}$ of n distinct positive integers such that $a_1 \cdots a_n$ divides $(x + a_1) \cdots (x + a_n)$ for every positive integer x.

(Ankan Bhattacharya)

Answer: $\{a_1 \dots, a_n\} = \{1, \dots, n\}$. This works since

$$\frac{(x+n)\dots(x+1)}{n!} = \binom{x+n}{n} \in \mathbb{Z}$$

so we now show that it is the only possibility. There are two approaches.

First solution Let $P(x) = (x + a_1) \dots (x + a_n)$. Then, $a_1 \dots a_n$ should divide the nth finite difference of P, which is $n!$. But

$$a_1 \dots a_n \mid n! \implies \{a_1 \dots, a_n\} = \{1, \dots, n\}$$

for size reasons.

Second solution (Kevin Sun) Let $s + 1$ be the smallest positive integer not in our set A and denote $B = A \setminus \{1, \dots, s\}$.

It's clear that the divisibility holds for negative x as well. Set $x = -s - 1$ to obtain

$$\begin{aligned}
\mathbb{Z} \ni \frac{1}{a_1 \dots a_n} \prod_{a \in A} (x + a) \\
&= \prod_{a \in A} \left(1 + \frac{x}{a}\right) \\
&= \prod_{a \in \{1,\dots,s\}} \left(1 - \frac{s+1}{a}\right) \cdot \prod_{b \in B} \left(1 - \frac{s+1}{b}\right) \\
&= \prod_{a \in \{1,\dots,s\}} \left(\frac{a - (s+1)}{a}\right) \cdot \prod_{b \in B} \left(1 - \frac{s+1}{b}\right) \\
&= \frac{(-s)(-(s-1))\dots(-1)}{1 \cdot 2 \cdot \dots \cdot s} \cdot \prod_{b \in B} \left(1 - \frac{s+1}{b}\right) \\
&= (-1)^{|A|} \prod_{b \in B} \left(1 - \frac{s+1}{b}\right).
\end{aligned}$$

If B is nonempty this has magnitude strictly between 0 and 1, (since $\min B > s + 1$ and thus each term is in $(0, 1)$). Thus B is empty and $A = \{1, \dots, s\}$.

N2. Call a number n *good* if it can be expressed in the form $2^x + y^2$ where x and y are nonnegative integers.

(a) Prove that there exist infinitely many sets of 4 consecutive good numbers.

(b) Find all sets of 5 consecutive good numbers.

(Michael Ma)

For (a), note that for any t, the numbers $t^2 + 1$, $t^2 + 2$, $t^2 + 4$ are good. So it suffices to show $t^2 + 3$ is good infinitely often, that is, $t^2 + 3 = 2^x + y^2$ has infinitely many nonnegative integer solutions (since for fixed t there are finitely many (x, y)). But this rearranges $t^2 - y^2 = 2^x - 3$ which has a solution for every x.

We now turn to the laborious task of (b), determining all sets of five consecutive good numbers. The answers are the six tuples $\{1, 2, 3, 4, 5\}$, $\{2, 3, 4, 5, 6\}$, $\{8, 9, 10, 11, 12\}$, $\{9, 10, 11, 12, 13\}$, $\{288, 289, 290, 291, 292\}$, $\{289, 290, 291, 292, 293\}$. These all work since

$$\begin{gathered} 1 = 2^0 + 0^2, \quad 2 = 2^0 + 1^2, \quad 3 = 2^1 + 1^2, \\ 4 = 2^2 + 0^2, \quad 5 = 2^2 + 1^2, \quad 6 = 2^1 + 2^2 \\ 8 = 2^3 + 0^2, \quad 9 = 2^3 + 1^2, \quad 10 = 2^0 + 3^2, \\ 11 = 2^1 + 3^2, \quad 12 = 2^3 + 2^2, \quad 13 = 2^2 + 3^2, \\ 288 = 2^5 + 16^2, \quad 289 = 2^6 + 15^2, \quad 290 = 2^0 + 17^2, \\ 291 = 2^1 + 17^2, \quad 292 = 2^8 + 6^2, \quad 293 = 2^2 + 17^2. \end{gathered}$$

We now show they are the only ones. First, consider the following table which shows $2^x + y^2 \pmod 8$:

	$x = 0$	$x = 1$	$x = 2$	$x \geq 3$
$y \equiv 1 \pmod 2$	2	3	5	1
$y \equiv 0 \pmod 4$	1	2	4	0
$y \equiv 2 \pmod 4$	5	6	0	4

Note that from this table, no good number is $7 \pmod 8$. Thus any five good numbers must have a $3 \pmod 8$ number. By table can only occur if that good number is of the form $t^2 + 2^1 = t^2 + 2$ for an odd integer t.

We now have several cases.

Case 1: Suppose the five good numbers are $\{t^2 + 1, t^2 + 2, t^2 + 3, t^2 + 4, t^2 + 5\}$.

Note that $t^2 + 5 \equiv 6 \pmod 8$, and by table, this can only occur if $t^2 + 5 = s^2 + 2^2 = s^2 + 4$ for some integer s; hence $t^2 - s^2 = 1$, so $t = 1$ and $s = 0$. This gives the solution set $\{2, 3, 4, 5, 6\}$.

Case 2: Suppose the five good numbers are $\{t^2, t^2 + 1, t^2 + 2, t^2 + 3, t^2 + 4\}$.

Since t^2 is good, we have $t^2 = 2^w + z^2$ for some w and z, which we write as $(t - z)(t + z) = 2^w$.

We now split into cases.

- **Subcase 2.1**: We handle the situation where $w < 4$.
  - If $w = 0$, then we get $t = 1$, which gives the solution $\{1, 2, 3, 4, 5\}$.
  - If $w = 1$, then there are no solutions by taking mod 4.

  - If $w = 2$, then $t^2 = 4 + z^2$ which implies $t = 2$, but t was odd.
  - If $w = 3$, we get $t^2 = 8 + z^2$ which implies $t = 3$, which gives $\{9, 10, 11, 12, 13\}$.
  - If $w = 4$, we get $t^2 = 16 + z^2$ which together with t odd implies $t = 5$, which gives $\{25, 26, 27, 28, 29\}$. However, the number 28 is not good, so this is not a solution.

- **Subcase 2.2**: Suppose $w \geq 5$. As $\gcd(t - z, t + z) \mid 2t$ we must have $t - z = 2$, $t + z = 2^{w-1}$, and thus $t = \frac{1}{2}\left(2 + 2^{w-1}\right) = 2^{w-2} + 1$. Since t was odd, we actually have $w \geq 3$.

  But $t^2 + 3$ is also good, so write

  $$t^2 + 3 = 2^x + y^2.$$

  So we split into cases again.

  - **Subcase 2.2.1**: We handle the case $x < 3$.
    * If $x = 0$, we get $t^2 + 2 = y^2$ which has no solutions.
    * If $x = 1$, we get $t^2 + 1 = y^2$ which implies $t = 0$, but t is supposed to be odd.
    * If $x = 2$, then we get $t^2 = y^2 + 1$ which implies $t = 1$, which was an earlier solution.
  - **Subcase 2.2.2**: Otherwise, assume $x \geq 3$.

    $$\begin{aligned} 2^x + y^2 &= t^2 + 3 \\ \implies 2^x + y^2 &= \left(2^{w-2} + 1\right)^2 + 3 \\ &= 2^{2w-4} + 2^{w-1} + 4 \\ \implies 2^{2w-6} + 2^{w-3} + 1 &= 2^{x-2} + (y/2)^2 \end{aligned}$$

    since y is clearly even; the last line implies $y/2$ is odd, since $2w - 6 > 0$, $w - 3 > 0$, $x - 2 > 0$.

    Let $c = w - 3 \geq 2$, $a = x - 2 \geq 1$, $b = y/2 \geq 1$ for brevity; then the equation rewrites as

    $$2^{2c} + 2^c + 1 = 2^a + b^2.$$

    We rewrite this as

    $$(2^c + 1 - b)(2^c + 1 + b) = (2^c + 1)^2 - b^2 = 2^a + 2^c \geq 0.$$

    In light of this, we have $2^a + 2^c \geq (2^c + 1)^2 - 2^{2c} > 2^{c+1}$, so $2^a > 2^c$, ergo $a > c$. Thus we may further write

    $$(2^c + 1 - b)(2^c + 1 + b) = 2^c(2^{a-c} + 1).$$

    The factors on the left-hand side are nonnegative and have gcd dividing $2b$, hence one of them has at most one factor of 2. So one of the factors must be divisible by 2^{c-1}. Thus, $b \equiv \pm 1 \pmod{2^{c-1}}$.

    But, $b < 2^c + 1$. So we have four possibilities:

    * **Subcase 2.2.2.1**: suppose $b = 1$. Then we get $2^{2c} + 2^c = 2^a$, which is impossible.

* **Subcase 2.2.2.2**: suppose $b = 2^{c-1} - 1$. Then we get $(2^{c-1} + 2)(2^c + 2^{c-1}) = 2^c(2^{a-c} + 1)$ and hence $3 \cdot 2^{c-2} = 2^{a-c} - 2$. This implies $a - c = 3$ and $c - 2 = 1$, so $c = 3$, or $w = 6$, hence $t = 2^{w-2} + 1 = 17$.

  This gives $\{289, 290, 291, 292, 293\}$ which indeed works.

* **Subcase 2.2.2.3**: suppose $b = 2^{c-1}+1$. Then we get $2^{c-1}(2^c+2^{c-1}+2) = 2^c(2^{a-c} + 1)$, or $2^{c-1} + 2^{c-2} + 1 = 2^{a-c} + 1$, which is impossible.

* **Subcase 2.2.2.4**: suppose $b = 2^c - 1$. This gives $2 \cdot 2^{c+1} = 2^c(2^{a-c} + 1)$, which is impossible.

Case 3: Suppose the five good numbers are $\{t^2 - 1, t^2, t^2 + 1, t^2 + 2, t^2 + 3\}$.

In that case, $\{t^2, t^2 + 1, t^2 + 2, t^2 + 3, t^2 + 4\}$ is also a set of five consecutive good numbers. Using case 2, the new candidate this now gives are $\{8, 9, 10, 11, 12\}$ and $\{288, 289, 290, 291, 292\}$, which work.

N3. Let $a_1, a_2, \ldots$ be an infinite sequence of positive integers satisfying $a_1 = 1$ and

$$a_n \mid a_k + a_{k+1} + \cdots + a_{k+n-1}$$

for all positive integers k and n. Find the maximum possible value of a_{2018}.

(Krit Boonsiriseth)

The answer is $a_{2018} \le 2^{1009} - 1$. To see this is attainable, consider the sequence

$$a_n = \begin{cases} 1 & n \text{ odd} \\ 2^{n/2} - 1 & n \text{ even.} \end{cases}$$

This can be checked to work, so we prove it's optimal.

We have $a_2 \mid a_1 + a_2 = 1 + a_2 \implies a_2 = 1$.

Now consider an integer n, and let $s = s_n = a_1 + \cdots + a_n$. Then

$$\begin{aligned} a_{n+1} &\mid s \\ a_{n+2} &\mid s + a_{n+1} \\ a_{n+2} &\equiv 1 \pmod{a_{n+1}}. \end{aligned}$$

Thus, $\gcd(a_{n+2}, a_{n+1}) = 1$. So $a_{n+2} \le \frac{s+a_{n+1}}{a_{n+1}}$, and thus

$$a_{n+1} + a_{n+2} \le 1 + a_{n+1} + \frac{s}{a_{n+2}} \le s + 2.$$

So, we have

$$\begin{aligned} a_1 + a_2 &= 2 \\ a_3 + a_4 &\le 2 + 2 = 4 \\ a_5 + a_6 &\le (2+4) + 2 = 8 \\ a_7 + a_8 &\le (2+4+8) + 2 = 16 \\ &\vdots \\ a_{2017} + a_{2018} &\le 2^{1009}. \end{aligned}$$

Thus $a_{2018} \le 2^{1009} - a_{2017} \le 2^{1009} - 1$.

Remark (Motivational notes)**.** It's very quick to notice $a_{n+1} \mid a_1 + \cdots + a_n$, which already means that given the first n terms of the sequence there are finitely many possibilities for the next one. Thus it's possible to play with "small cases" by drawing a large tree.

When doing so, one might hope that somehow $a_n = a_1 + \cdots + a_{n-1}$ is achievable, but quickly notices in such a tree that if a_n is the sum of all previous terms, then $a_{n+1} = 1$ is forced. This gives the idea to try to look at the terms in pairs, rather than one at a time, and this gives the correct bound.

As for extracting the equality case from this argument, there are actually two natural curves to try. We have $a_3 \mid 1+1 = 2$. If we have $a_3 = 2$ we get $a_4 = 1$, $a_5 \le 5$, but then a_6 actually gets stuck. But if we have $a_3 = 1$ instead, we get $a_4 = 3$, $a_5 = 1$, $a_6 = 7$, and so on; pushing this gives the equality case above, seen to work. I think it's quite unnatural to guess the correct construction before having the corresponding $s + 2$ estimate.

N4. Fix a positive integer $n > 1$. We say a nonempty subset S of $\{0, 1, \dots, n-1\}$ is d-coverable if there exists a polynomial P with integer coefficients and degree at most d, such that S is exactly the set of residues modulo n that P attains as it ranges over the integers.

For each n, determine the smallest d such that any nonempty subset of $\{0, \dots, n-1\}$ is d-coverable, or prove that no such d exists.

(Carl Schildkraut)

This is possible for $n = 4$ or n prime, in which case $d = n - 1$ is best possible. Let $P(\mathbb{Z}/n)$ denote the range of a polynomial modulo n.

- We first note that if $n = q_1 \dots q_k$ is the product of $k \ge 2$ distinct prime powers, then
$$|P(\mathbb{Z}/n)| = \prod_{i=1}^{k} |P(\mathbb{Z}/q_i)|.$$
Hence any subset S with size $n - 1$ is not coverable.

- If $n = p^e$ is a prime power with other than 4 with $e \ge 2$, consider the set $S = \{0, 1, \dots, p-1, p\}$. We claim it is not coverable.

  Indeed, if P covers it, WLOG $P(0) = 0$. Now, P is surjective modulo p, hence bijective, and thus $P(x) \equiv 0 \pmod p \iff x \equiv 0 \pmod p$. Now we can write
$$P(x) = a_1 x + a_2 x^2 + \dots.$$
  - If $a_1 \equiv 0 \pmod p$, then $x \equiv 0 \pmod p \implies P(x) \equiv 0 \pmod{p^2}$, so p does not appear in the image.
  - If $a_1 \not\equiv 0 \pmod p$, then p, $2p$, $\dots$ all appear in the miage, which is wrong for $n > 4$.

- Let $n = 4$, and consider $S \pmod 4$.
  - If $S = \{k\}$ take $P(x) = k$.
  - If $S = \{k, k+1\}$ take $P(x) = x^2 + k$.
  - If $S = \{k, k+2\}$ take $P(x) = 2x^2 + k$.
  - If $S = \{k-1, k, k+1\}$ take $P(x) = x^3 + k$.

    We claim also the example $S = \{-1, 0, 1\}$ is not 2-coverable. Indeed, WLOG $P(0) = 0$ so $P(x) = x(x + c)$. Then $P(2) \equiv 0 \pmod 4$, meaning c is even. But then $P(1) \equiv c + 1 \pmod 4$ and $P(-1) \equiv 1 - c \pmod 4$, so $P(1) \equiv P(-1)$.
  - If $S = \{0, 1, 2, 3\}$ take $P(x) = x$.

- Let $n = 2$.
  - If $S = \{k\}$ take $P(x) = k$.
  - If $S = \{0, 1\}$ take $P(x) = x$. This is obviously not 0-coverable.

- If $n = p$ is an odd prime, we claim $S = \{1, \dots, p-1\}$ is not $(p-2)$-coverable. Indeed, suppose $P(x) = a_{p-2}x^{p-2} + \dots + a_0$ covered it. Then

$$\sum_x P(x) \equiv \sum_k a_k \sum_x x^k \equiv 0 \pmod p.$$

However, if $P(\mathbb{Z}/p) = \{1, \dots, p-1\}$ then some element appears twice and the others appear once. If k is the repeated element though, then $\sum_x P(x) = (1 + \dots + (p-1)) + k \equiv k \not\equiv 0 \pmod p$.