Elmo Lives Mostly Outside

Year: **2016**

18th ELMO
Pittsburgh, PA

Day: **1**

Saturday, June 18, 2016
1:15PM — 5:45PM

Problem 1. Cookie Monster says a positive integer n is *crunchy* if there exist $2n$ real numbers $x_1, x_2, \ldots, x_{2n}$, not all equal, such that the sum of any n of the x_i's is equal to the product of the other n of the x_i's. Help Cookie Monster determine all crunchy integers.

Problem 2. Oscar is drawing diagrams with trash can lids and sticks. He draws a triangle ABC and a point D such that DB and DC are tangent to the circumcircle of ABC. Let B' be the reflection of B over AC and C' be the reflection of C over AB. If O is the circumcenter of $DB'C'$, help Oscar prove that AO is perpendicular to BC.

Problem 3. In a Cartesian coordinate plane, call a rectangle *standard* if all of its sides are parallel to the x- and y- axes, and call a set of points *nice* if no two of them have the same x- or y- coordinates. First, Bert chooses a nice set B of 2016 points in the coordinate plane. To mess with Bert, Ernie then chooses a set E of n points in the coordinate plane such that $B \cup E$ is a nice set with $2016 + n$ points. Bert returns and then miraculously notices that there does not exist a standard rectangle that contains at least two points in B and no points in E in its interior. For a given nice set B that Bert chooses, define $f(B)$ as the smallest positive integer n such that Ernie can find a nice set E of size n with the aforementioned properties. Help Bert determine the minimum and maximum possible values of $f(B)$.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

Sunday, June 19, 2016
1:15PM — 5:45PM

Problem 4. Big Bird has a polynomial P with integer coefficients such that n divides $P(2^n)$ for every positive integer n. Prove that Big Bird's polynomial must be the zero polynomial.

Problem 5. Elmo is drawing with colored chalk on a sidewalk outside. He first marks a set S of $n > 1$ collinear points. Then, for every unordered pair of points $\{X, Y\}$ in S, Elmo draws the circle with diameter XY so that each pair of circles which intersect at two distinct points are drawn in different colors. Count von Count then wishes to count the number of colors Elmo used. In terms of n, what is the minimum number of colors Elmo could have used?

Problem 6. Elmo is now learning olympiad geometry. In a triangle ABC with $AB \neq AC$, let its incircle be tangent to sides BC, CA, and AB at D, E, and F, respectively. The internal angle bisector of $\angle BAC$ intersects lines DE and DF at X and Y, respectively. Let S and T be distinct points on side BC such that $\angle XSY = \angle XTY = 90°$. Finally, let γ be the circumcircle of $\triangle AST$.

(a) Help Elmo show that γ is tangent to the circumcircle of $\triangle ABC$.

(b) Help Elmo show that γ is also tangent to the incircle of $\triangle ABC$.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.