**$18^{\text{th}}$ Elmo Lives Mostly Outside**

**ELMO 2016**

**Pittsburgh, PA**

**OFFICIAL SOLUTIONS**

1. Cookie Monster says a positive integer $n$ is *crunchy* if there exist $2n$ real numbers $x_1, x_2, \ldots, x_{2n}$, not all equal, such that the sum of any $n$ of the $x_i$'s is equal to the product of the other $n$ of the $x_i$'s. Help Cookie Monster determine all crunchy integers.

   *Proposed by Yannick Yao.*

   **Answer.** The crunchy numbers are exactly the even integers $n = 2, 4, 6, \ldots$.

   **Solution.** Notice that

$$\prod_{i=1}^{2n} x_n = (x_{a_1} + x_{a_2} + \ldots + x_{a_n})(x_{a_{n+1}} + x_{a_{n+2}} + \ldots + x_{a_{2n}})$$

   where the $a_i$ are any permutation of $1 - 2n$. Switching $a_n$ and $a_{n+1}$ in the formula and setting both sides to be equal we get an equation that factors into

$$(x_{a_n} - x_{a_{n+1}})\left[(x_{a_1} + x_{a_2} + \ldots + x_{a_{n-1}}) - (x_{a_{n+2}} + x_{a_{n+3}} + \ldots + x_{a_{2n}})\right] = 0.$$

   Since not all of the numbers are equal we can see that if any two are not equal then the other $2n - 2$ must be equal by permuting $a_i$ in the above equation. Also one of these two must share the same value as these $2n - 2$ by the same logic. So WLOG $x_1 = x_2 = \ldots = x_{2n-1} = x$ and $a_{2n} = y$. So we end up with the equations

$$nx = x^{n-1}y \qquad (n-1)x + y = x^n.$$

   Notice $x \neq 0$ or else $y$ would also be 0. Substituting $y = \frac{n}{x^{n-2}}$ into the second equation, clearing denominators, and factoring gives us

$$(x^{n-1} - n)(x^{n-1} + 1) = 0.$$

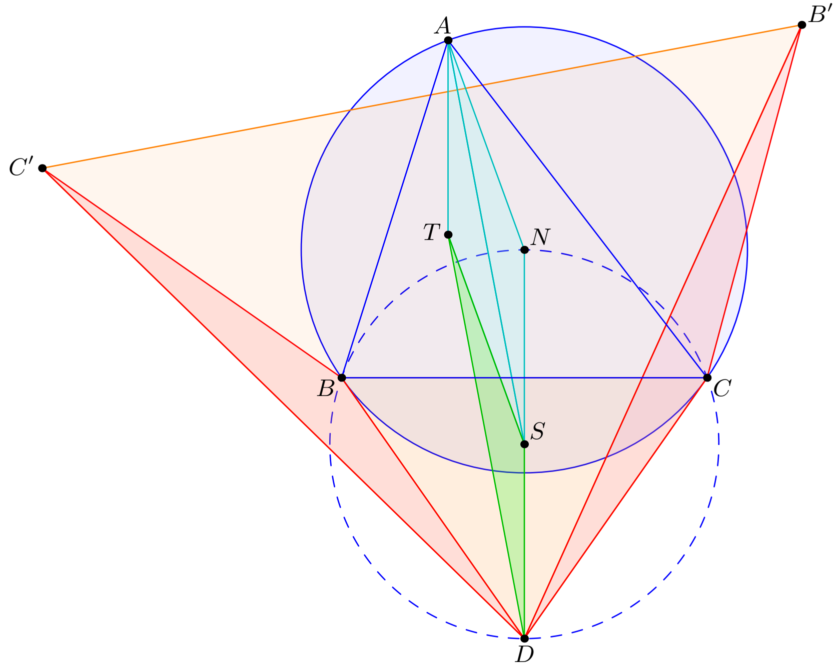
   If $x = \sqrt[n-1]{n}$ then $y$ would also be $\sqrt[n-1]{n}$. Thus, $n - 1$ must be odd and then $n$ must be even. Say $n$ is even. Then setting $x_1 = x_2 = \ldots = x_{2n-1} = -1$ and $x_{2n} = n$ clearly works and we are done ■

   This problem was proposed by Yannick Yao. This solution was given by Michael Ma.

2. Oscar is drawing diagrams with trash can lids and sticks. He draws a triangle $ABC$ and a point $D$ such that $DB$ and $DC$ are tangent to the circumcircle of $ABC$. Let $B'$ be the reflection of $B$ over $AC$ and $C'$ be the reflection of $C$ over $AB$. If $O$ is the circumcenter of $DB'C'$, help Oscar prove that $AO$ is perpendicular to $BC$.

*Proposed by James Lin.*

**Solution 1.** Let $N$ denote the circumcenter of $ABC$ and let $S$ denote the circumcenter of $NBDC$ (midpoint of $\overline{ND}$). Let $T$ be the point such that $ANST$ is a parallelogram (hence $ASDT$ too). We will prove that $T = O$, which implies the result (since $\overline{NSD} \perp \overline{BC}$).




First we claim that $\triangle B'CD \sim \triangle TSD$, with equal orientation. By angle chasing, we have

$$\begin{aligned} \measuredangle TSD = \measuredangle ANS &= \measuredangle(\overline{AN}, \overline{BC}) + 90^\circ = (\measuredangle NAC + \measuredangle ACB) + 90^\circ \\ &= (90^\circ - \measuredangle CBA) + \measuredangle ACB + 90^\circ = 2\measuredangle ACB + \measuredangle BAC \\ &= \measuredangle B'CA + \measuredangle ACB + \measuredangle BCD = \measuredangle B'CD. \end{aligned}$$

Finally from isosceles $\triangle DBC \sim \triangle SBN$, we have

$$\frac{B'C}{CD} = \frac{BC}{CD} = \frac{BN}{NS} = \frac{NA}{NS} = \frac{TS}{SD}.$$

This implies the similarity.

Similarly, $\triangle C'BD \sim \triangle TSD$. Then there is a spiral similarity sending $\triangle DBC$ to $\triangle DB'C'$, and sending $S$ to $T$. As $S$ is the circumcenter of $\triangle DBC$, $T$ is the circumcenter of $\triangle DB'C'$, meaning $T = O$. ■

This first solution was suggested by Evan Chen.

**Solution 2.** First, note that triangles $DBC'$ and $DCB'$ are congruent and in the same orientation, so $DB'C'$ is similar to $DBC$. Now, let the circumcircle of $DB'C'$ intersect $DB$ at $P$ and $DC$ at $Q$. We have that $\angle C'PB = \angle C'PD = \angle C'B'D =$

$\angle CBD = \angle BAC = \angle C'AB$, so $P$ lies on the circumcircle of $ABC'$. Furthermore, $\angle ABP = \angle ACB = \angle AC'B = \angle APB$, so $AP = AB$. Similarly, $AQ = AC$. Now, let $X$ and $Y$ be on $DB$ and $DC$ so that $AD = AX = AY$.

The key lemma is that given varying points $D$ and $E$ on fixed rays $AB$ and $AC$ such that $AD - AE$ is constant. Then the circumcenter of $ADE$ lies on a fixed line parallel to the angle bisector of $\angle BAC$. The proof of this is that all circumcircles of $ADE$ share a common midpoint of arc $DAE$, call it $Z$, by spiral similarity, so the circumcenter of $ADE$ lies on the perpendicular bisector of $AZ$, which is a fixed line parallel to the angle bisector.

Now, we use this lemma on rays $DB$ and $DC$. Note that since triangles $ADX$, $ABP$, $ADY$, and $ACQ$ are all isosceles, $DX - DP = XP = DB = DC = YQ = DY - DQ$, so we have that $DX - DY = DP - DQ$. Now, note that the circumcenter of $DPQ$ is $O$ and the circumcenter of $DXY$ is $A$, so the line through them is perpendicular to $BC$ by the lemma, as desired. ■

This second solution was suggested by Michael Ren.

This problem was proposed by James Lin.

3. In a Cartesian coordinate plane, call a rectangle *standard* if all of its sides are parallel to the $x$- and $y$- axes, and call a set of points *nice* if no two of them have the same $x$- or $y$- coordinates. First, Bert chooses a nice set $B$ of 2016 points in the coordinate plane. To mess with Bert, Ernie then chooses a set $E$ of $n$ points in the coordinate plane such that $B \cup E$ is a nice set with $2016 + n$ points. Bert returns and then miraculously notices that there does not exist a standard rectangle that contains at least two points in $B$ and no points in $E$ in its interior. For a given nice set $B$ that Bert chooses, define $f(B)$ as the smallest positive integer $n$ such that Ernie can find a nice set $E$ of size $n$ with the aforementioned properties. Help Bert determine the minimum and maximum possible values of $f(B)$.

   *Proposed by Yannick Yao.*

   **Solution 1.** The minimum is 2015, since there needs to be a point in $J$ whose $x$-coordinate is between each two consecutive points in $A$ when sorted by $x$-coordinate. The minimum is achieved when $A = \{(t,t) | t = 0, 1, \cdots, 2015\}$.

   For general $|A| = c$ (instead of 2016) the maximum is $2c - 2\sqrt{c}$

   To keep things clean, I will let $c = k^2$ where $k$ is a positive integer. The construction, as mentioned above is to take a $k$ by $k$ square and rotate it slightly.

   Now to show that $2c - 2k$ suffices, consider the set of points in $A$ as a poset where for points $p, q$, $p > q$ if $p$ is up and right of $q$.

   Take the longest antichain and say it has $s$ elements. This antichain is actually an up left chain of points. Partition the remaining points into two sets, those that are $>$ than some element in the antichain and those that are $<$ some element in the antichain. For the first set, Ernie draws points slightly below and left of each point and Ernie draws points slightly above and right of each point in the second set.

In total Ernie has drawn $k^2 - s$ points. (We have eliminated all possible rectangles where the two points in A form an up right vector since these two points cannot both be in the antichain)

Now we can do the same for up left rectangles. To finish the problem it suffices to note Dilworth's theorem and use AM-GM. ■

This first solution was suggested by Allen Liu.

**Solution 2.** Here is an alternative way to show the maximum. As above, the number of points needed for $J$ is equal to

- Twice the number of points,
- minus the length of the maximal down-right chain, and
- minus the length of the maximal up-right chain.

If we order the points by their $x$-coordinate and consider a sequence being their $y$-coordinates, the two things we are subtracting becomes the length of maximal decreasing subsequence and the length of maximal increasing subsequence respectively. Notice that if the two lengths are $m$ and $n$ respectively, then the number of points is at most $mn$, because of the famous result that a sequence of $mn+1$ distinct real numbers must either contain an increasing subsequence of length $m+1$ or a decreasing subsequence of length $n+1$.

Therefore, in the context of this particular case, we have $mn \geq 2016$ and we need to maximize $2 \cdot 2016 - m - n$, and this is easy by AM-GM, and the maximized result is $2 \cdot 2016 - \lceil 2\sqrt{2016} \rceil = 3942$.

This maximum is achieved by having a slightly tilted $42 \times 48$ lattice grid for $A$. ■

4. Big Bird has a polynomial $P$ with integer coefficients such that $n$ divides $P(2^n)$ for every positive integer $n$. Prove that Big Bird's polynomial must be the zero polynomial.

   *Proposed by Ashwin Sah.*

   **Solution.** We claim $P(2^k) = 0$ for every positive integer $k$, which is enough. Indeed, for $p$ prime we have
   $$0 \equiv P(2^{kp}) \equiv P(2^k) \pmod{p}$$
   since $2^{kp} \equiv 2^k \pmod{p}$, so the claim follows by taking $p$ sufficiently large. ■

   This problem and solution were proposed by Ashwin Sah.

5. Elmo is drawing with colored chalk on a sidewalk outside. He first marks a set $S$ of $n > 1$ collinear points. Then, for every unordered pair of points $\{X, Y\}$ in $S$, Elmo draws the circle with diameter $XY$ so that each pair of circles which intersect at two distinct points are drawn in different colors. Count von Count then wishes to count the number of colors Elmo used. In terms of $n$, what is the minimum number of colors Elmo could have used?

   *Proposed by Michael Ren.*

**Answer.** The answer is $\lceil n/2 \rceil$ colors, except when $n = 3$ here the answer is 1.

**Solution.** I claim that the answer for even $n$ is $n/2$. We can let the distance between adjacent points be 1. Label the vertices $1, 2, 3, \ldots, n/2, 1, 2, \ldots, n/2$ in that order, left to right (we can assume that the line the points are on is horizontal).

Now, consider the $n/2$ circles whose diameters have endpoints with the same label. Note that these are pairwise intersecting, so we must use at least $n/2$ colors.

For the coloring, for circles with diameter $\leq n/2$, color them with the label of the right endpoint of the diameter. For circles with diameter $\geq n/2$, color them with the label of the left endpoint of the diameter. By checking cases, it is not hard to confirm that this coloring works.

Now we consider $n = 2m + 1$ odd. Obviously $n = 3$ gives 1. For other $n = 2m + 1$ we will show the minimum number of colors is $m + 1$. We can construct this by using the above construction, and coloring each circle containing $2m + 1$ with the color $m + 1$.

Now, for proving it, call the vertices $1, 2, \ldots, m, m + 1, 1, 2, \ldots, m$ as earlier; we have the $m$ different colored circles from vertices of the same color. Let $f(a, b)$ denote the color of the circle with vertex color $a$ in the first $m + 1$ vertices, and vertex color $b$ in the last $m + 1$ vertices. Note that $f(1, m + 1) = 1$. $f(2, 1) = 1, 2$, but we know it is 2 due to the previous conclusion. Similarly, we show that $f(k + 1, k) = k + 1$ for $1 \leq k \leq m$, so in particular, we need $f(m + 1, m) = m + 1$, as desired. ■

This problem was proposed by Michael Ren. This solution was given by Mihir Singhal and James Lin.

**Remark.** An alternate easier version of the problem requires that circles which are tangent to each other are also distinct colors. In this case the answer is $n$.

Label the vertices $1, 2, \ldots n$, and let $f(a, b)$ be the circle with diameter at vertices $a, b$. Note that $f(1, 2), f(1, 3), \ldots, f(1, n), f(2, n)$ are different colors, so at least $n$ colors are needed. But then we can let $f(a, b)$ be colored by color $a + b \pmod n$, so we are done!

6. Elmo is now learning olympiad geometry. In a triangle $ABC$ with $AB \neq AC$, let its incircle be tangent to sides $BC$, $CA$, and $AB$ at $D$, $E$, and $F$, respectively. The internal angle bisector of $\angle BAC$ intersects lines $DE$ and $DF$ at $X$ and $Y$, respectively. Let $S$ and $T$ be distinct points on side $BC$ such that $\angle XSY = \angle XTY = 90^\circ$. Finally, let $\gamma$ be the circumcircle of $\triangle AST$.

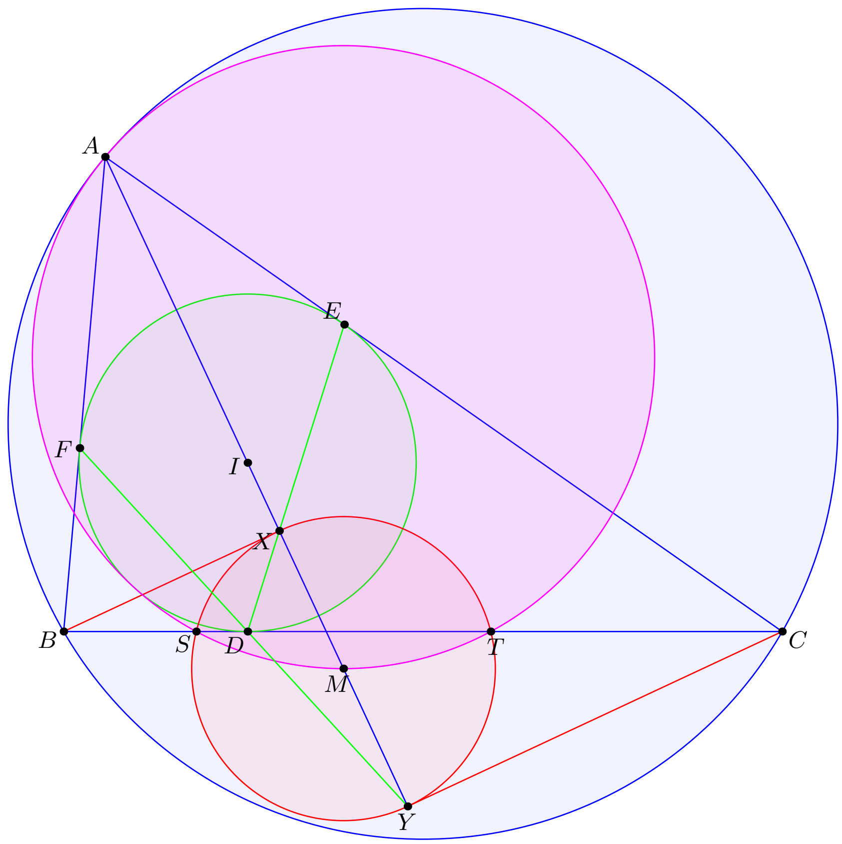
   (a) Help Elmo show that $\gamma$ is tangent to the circumcircle of $\triangle ABC$.

   (b) Help Elmo show that $\gamma$ is also tangent to the incircle of $\triangle ABC$.

   *Proposed by James Lin.*

**Solution 1.** First, we claim that $X$ and $Y$ are the incenter and excenter of $\triangle AST$. (This is Sharygin 2013, Problem 18, also problem 11.12 of Euclidean Geometry in Mathematical Olympiads.) To see this, recall that $\angle AXB = \angle AYC$ are right angles (see for example JMO 2014 problem 6). Now let $K = \overline{AXY} \cap \overline{BC}$ and let $L$ be the foot of the external $\angle A$-bisector. Then $(KL; BC) = -1$, so projection onto $\overline{AI}$ gives

$(AK;XY) = -1$. Now, since $\angle YSX = 90^\circ$, we see that $\overline{SX}$ and $\overline{SY}$ are bisectors of $\angle AST$. The same statement holds for $\angle ATS$, which proves the claim.




In particular, this implies that $\overline{AS}$ and $\overline{AT}$ are isogonal to each other, and therefore part (a) is solved.

As for part (b), denote $(XSTY)$ by $\omega$, centered at a point $M$, which is midpoint of arc $ST$ of $\gamma$. Now, we observe that $\triangle IXD \sim \triangle IDY$, therefore $ID^2 = IX \cdot IY$ and thus the incircle is orthogonal to $\omega$. Therefore an inversion around $\omega$ fixes the incircle. Now $\gamma$ is mapped to line $BC$, which is obviously tangent to incircle. Therefore $\gamma$ was tangent too. ■

This first solution was suggested by Evan Chen.

**Solution 2.** Here is an alternate solution to part (b).

Let the $A$-excircle of $ABC$ be tangent to $AB$ at $R$ and $BC$ at $S$. It is well-known that $X$ lies on $RS$ and $Y$ lies on $DE$. Hence, by some angle-chasing $ARX$ and $AYE$ are similar (both have angles $\frac{\angle A}{2}, \frac{\angle B}{2}, 90 + \frac{\angle C}{2}$), so we have that $AR \cdot AE = AX \cdot AY = AS \cdot AT$. Hence, a $\sqrt{bc}$ inversion on $AST$ swaps the incircle and $A$-excircle of $ABC$. But it also swaps the circumcircle of $AST$ and $ST$. Since the incircle and $A$-excircle of $ABC$ are both tangent to $ST$, or $BC$, both are also tangent to the circumcircle of $AST$, as desired. ■

This second solution was suggested by Michael Ren..

**Solution 3.** We also claim $(AST)$ is tangent to the $A$-excircle.

It's well-known and you can prove with angle-chasing that $X, Y$ are the feet of the perpendiculars from $B, C$ to $AI$, where $I$ is the incenter.

Let $M$ be the midpoint of $BC$ and $N$ be the midpoint of $XY$. Clearly $MN \perp XY$ so we get that $N$ lies on the radical axis of the incircle and $A$-excircle, and it is obvious that $N$ is the center of the circle of diameter $XY$.

Note that $IX \perp BX$. Let $B'$ be the midpoint of $DF$, so that $B, B'$ correspond in inversion about the incircle. Thus, if $X'$ is the image of $X$ under inversion about the incircle, we should have that $\angle IB'X' = 90^\circ$ so that $X'$ lies on $DF$. Then it's clear that $X' = Y$ so $X, Y$ are inverses under inversion about the incircle.

Now this means that $(XY)$ is orthogonal to the incircle. Note that since $N$ is on the radical axis of the incircle and $A$-excircle, $P(N, \text{incircle}) = P(N, \text{A-excircle}) = NX$ which means $(XY)$ is also orthogonal to the $A$-excircle.

Now let $Z$ be the foot of the $A$-angle bisector. We claim that $(AZ), (XY)$ are orthogonal. It suffices to show $(A, Z; X, Y)$ is harmonic. Let $Z'$ be the foot of the $A$-external angle bisector. Project $(A, Z; X, Y)$ from $\infty_{AZ'}$ down to line $BC$ so it follows that $(A, Z; X, Y) = (Z', Z; B, C)$ which is clearly harmonic. Then $(AZ), (XY)$ are orthogonal as claimed. But then it follows that $A, Z$ are also inverses in inversion about circle $(XY)$.

Now invert $(AST)$ about $(XY)$. Clearly $S, T$ remain fixed while $A$ goes to $Z$ so $(AST)$ and line $BC$ are inverses. This can only happen if $(AST)$ passes through $N$, the center of inversion. Then $A, S, T, N$ are concyclic. Now it also follows from this inversion that since the incircle and excircle remain fixed, the image of $(AST)$ is tangent to both circles, so $(AST)$ was tangent to the incircle and $A$-excircle.

Now note that $N$ is the midpoint of arc $ST$ on $(AST)$ because $NS = NT$. But then it follows that $\angle SAN = \angle TAN$. Since $\angle BAN = \angle CAN$ we deduce that $AS, AT$ are isogonal w.r.t. $\angle BAC$.

Let $O_1$ be the circumcenter of $AST$ and $H$ be the orthocenter of $ABC$. Then $AH, AO$ are isogonal in triangle $ABC$ but $AH, AO_1$ are isogonal in triangle $AST$ so we deduce that $A, O, O_1$ are collinear. Then it follows that $(AST)$ is tangent to the circumcircle of $ABC$ as desired. ■

This third solution was suggested by Vincent Huang.

This problem was proposed by James Lin.