16^{th} Ego Loss May Occur

ELMO 2014
Lincoln, Nebraska

OFFICIAL SOLUTIONS

1. Find all triples (f, g, h) of injective functions from the set of real numbers to itself satisfying

$$\begin{aligned} f(x + f(y)) &= g(x) + h(y) \\ g(x + g(y)) &= h(x) + f(y) \\ h(x + h(y)) &= f(x) + g(y) \end{aligned}$$

for all real numbers x and y. (We say a function F is *injective* if $F(a) \neq F(b)$ for any distinct real numbers a and b.)

Proposed by Evan Chen.

Answer. For all real numbers x, $f(x) = g(x) = h(x) = x + C$, where C is an arbitrary real number.

Solution 1. Let a, b, c denote the values $f(0)$, $g(0)$ and $h(0)$. Notice that by putting $y = 0$, we can get that $f(x + a) = g(x) + c$, etc. In particular, we can write

$$h(y) = f(y - c) + b$$

and

$$g(x) = h(x - b) + a = f(x - b - c) + a + b$$

So the first equation can be rewritten as

$$f(x + f(y)) = f(x - b - c) + f(y - c) + a + 2b.$$

At this point, we may set $x = y - c - f(y)$ and cancel the resulting equal terms to obtain

$$f(y - f(y) - (b + 2c)) = -(a + 2b).$$

Since f is injective, this implies that $y - f(y) - (b+2c)$ is constant, so that $y - f(y)$ is constant. Thus, f is linear, and $f(y) = y + a$. Similarly, $g(x) = x + b$ and $h(x) = x + c$.

Finally, we just need to notice that upon placing $x = y = 0$ in all the equations, we get $2a = b + c$, $2b = c + a$ and $2c = a + b$, whence $a = b = c$.

So, the family of solutions is $f(x) = g(x) = h(x) = x + C$, where C is an arbitrary real. One can easily verify these solutions are valid. ■

This problem and solution were proposed by Evan Chen.

Remark. Although it may look intimidating, this is not a very hard problem. The basic idea is to view $f(0)$, $g(0)$ and $h(0)$ as constants, and write the first equation entirely in terms of $f(x)$, much like we would attempt to eliminate variables in a standard system of equations. At this point we still had two degrees of freedom, x and y, so it seems likely that the result would be easy to solve. Indeed, we simply select x in such a way that two of the terms cancel, and the rest is working out details.

Solution 2. First note that plugging $x = f(a), y = b; x = f(b), y = a$ into the first gives $g(f(a)) + h(b) = g(f(b)) + h(a) \implies g(f(a)) - h(a) = g(f(b)) - h(b)$. So $g(f(x)) = h(x) + a_1$ for a constant a_1. Similarly, $h(g(x)) = f(x) + a_2, f(h(x)) = g(x) + a_3$.

Now, we will show that $h(h(x)) - f(x)$ and $h(h(x)) - g(x)$ are both constant. For the second, just plug in $x = 0$ to the third equation. For the first, let $x = a_3, y = k$ in the original to get $g(f(h(k))) = h(a_3) + f(k)$. But $g(f(h(k))) = h(h(k)) + a_1$, so $h(h(k)) - f(k) = h(a_3) - a_1$ is constant as desired.

Now $f(x) - g(x)$ is constant, and by symmetry $g(x) - h(x)$ is also constant. Now let $g(x) = f(x) + p, h(x) = f(x) + q$. Then we get:

$$\begin{aligned} f(x + f(y)) &= f(x) + f(y) + p + q \\ f(x + f(y) + p) &= f(x) + f(y) + q - p \\ f(x + f(y) + q) &= f(x) + f(y) + p - q \end{aligned}$$

Now plugging in (x, y) and (y, x) into the first one gives $f(x + f(y)) = f(y + f(x)) \implies f(x) - x = f(y) - y$ from injectivity, $f(x) = x + c$. Plugging this in gives $2p = q, 2q = p, p + q = 0$ so $p = q = 0$ and $f(x) = x + c, g(x) = x + c, h(x) = x + c$ for a constant c are the only solutions. ■

This second solution was suggested by David Stoner.

Solution 3. By putting $(x, y) = (0, a)$ we derive that $f(f(a)) = g(0) + h(a)$ for each a, and the analogous counterparts for g and h. Thus we can derive from $(x, y) = (t, g(t))$ that

$$\begin{aligned} h(f(t) + h(g(t))) &= f(f(t)) + g(g(t)) \\ &= g(0) + h(t) + h(0) + f(t) \\ &= f(f(0)) + g(t + g(t)) \\ &= h(f(0) + h(t + g(t))) \end{aligned}$$

holds for all t. Thus by injectivity of h we derive that

$$f(x) + h(g(x)) = f(0) + h(x + g(x)) \quad (*)$$

holds for every x.

Now observe that placing $(x, y) = (g(a), a)$ gives

$$g(2g(a)) = g(g(a) + g(a)) = h(g(a)) + f(a)$$

while placing $(x, y) = (g(a) + a, 0)$ gives

$$g(g(a) + a + g(0)) = h(a + g(a)) + f(0).$$

Equating this via $(*)$ and applying injectivity of g again, we find that

$$2g(a) = g(a) + a + g(0)$$

for each a, whence $g(x) = x + b$ for some real number b. We can now proceed as in the earlier solutions. ■

This third solution was suggested by Mehtaab Sawhney.

Solution 4. In the first given, let $x = a + g(0)$ and $y = b$ to obtain

$$f(a + g(0) + f(b)) = g(a + g(0)) + h(b) = h(a) + h(b) + f(0).$$

Swapping the roles of a and b, we discover that

$$f(b + g(0) + f(a)) = f(a + g(0) + f(b)).$$

But f is injective; this implies $f(x) - x$ is constant, and we can the proceed as in the previous solutions. ■

This fourth solution was suggested by alibez.

2. Define a *beautiful number* to be an integer of the form a^n, where $a \in \{3,4,5,6\}$ and n is a positive integer. Prove that each integer greater than 2 can be expressed as the sum of pairwise distinct beautiful numbers.

Proposed by Matthew Babbitt.

Solution. First, we prove a lemma.

Lemma 1. *Let $a_0 > a_1 > a_2 > \cdots > a_n$ be positive integers such that $a_0 - a_n < a_1 + a_2 + \cdots + a_n$. Then for some $1 \le i \le n$, we have*

$$0 \le a_0 - (a_1 + a_2 + \cdots + a_i) < a_i.$$

Proof. Proceed by contradiction; suppose the inequalities are all false. Use induction to show that $a_0 - (a_1 + \cdots + a_i) \ge a_i$ for each i. This becomes a contradiction at $i = n$. □ □

Let N be the integer we want to express in this form. We will prove the result by strong induction on N. The base cases will be $3 \le N \le 10 = 6 + 3 + 1$.

Let $x_1 > x_2 > x_3 > x_4$ be the largest powers of $3, 4, 5, 6$ less than $N - 3$, in some order. If one of the inequalities of the form

$$3 \le N - (x_1 + \cdots + x_k) < x_k + 3; \quad 1 \le k \le 4$$

is true, then we are done, since we can subtract of $x_1, \ldots, x_k$ from N to get an N' with $3 \le N' < N$ and then apply the inductive hypothesis; the construction for N' cannot use any of $\{x_1, \ldots, x_k\}$ since $N' - x_k < 3$.

To see that this is indeed the case, first observe that $N - 3 > x_1$ by construction and compute

$$x_1 + x_2 + x_3 + x_4 + x_4 \ge (N-3) \cdot \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{6}\right) > N - 3.$$

So the hypothesis of the lemma applies with $a_0 = N - 3$ and $a_i = x_i$ for $1 \le i \le 4$.

Thus, we are done by induction. ■

This problem and solution were proposed by Matthew Babbitt.

Remark. While the approach of subtracting off large numbers and inducting is extremely natural, it is not immediately obvious that one should consider $3 \le N - (x_1 + \cdots + x_k) < x_k + 3$ rather than the stronger bound $3 \le N - (x_1 + \cdots + x_k) < x_k$. In particular, the solution method above does not work if one attempts to get the latter.

3. We say a finite set S of points in the plane is *very* if for every point X in S, there exists an inversion with center X mapping every point in S other than X to another point in S (possibly the same point).

   (a) Fix an integer n. Prove that if $n \ge 2$, then any line segment $\overline{AB}$ contains a unique very set S of size n such that $A, B \in S$.

   (b) Find the largest possible size of a very set not contained in any line.

(Here, an *inversion* with center O and radius r sends every point P other than O to the point P' along ray OP such that $OP \cdot OP' = r^2$.)

Proposed by Sammy Luo.

Answer. For part (b), the maximal size is 5.

Solution. For part (a), take a regular $(n+1)$-gon and number the vertices A_i $(i = 0, 1, 2, \ldots, n)$ Now invert the polygon with center A_0 with arbitrary power. This gives a very set of size n. (This can be easy checked with angle chase, PoP, etc.) By scaling and translation, this shows the existence of a very set as in part (a).

It remains to prove uniqueness. Suppose points $A = P_1, P_2, \ldots, P_n = B$ and $A = X_1, X_2, \ldots, X_n = B$ are two very sets on $\overline{AB}$ in that order. Assume without loss of generality that $X_1X_2 > P_1P_2$. Then $X_2X_1^2 = X_2X_3 \cdot (X_1X_n - X_1X_2) \implies X_2X_3 > P_2P_3$. Proceeding inductively, we find $X_kX_{k+1} > P_kP_{k+1}$ for $k = 1, 2, \ldots, n-1$. Thus, $X_1X_n > P_1P_n$, which is a contradiction.

For (b), let $P(A)$ (let's call this power, A is a point in space) be a function returning the radius of inversion with center A. Note that the power of endpoints of 1D very sets are equal, and these powers are the highest out of all points in the very set. Let the convex hull of our very set be H. Let the vertices be $A_1, A_2, ..., A_m$. (We have $m \geq 3$ since the points are not collinear.) Since A_1, A_2 are endpoints of a 1D very set, they have equal power. Going around the hull, all vertices have equal power.

Lemma 2. *Other than the vertices, no other points lie on the edges of H, and H is equilateral.*

Proof. Say X is on A_1A_2. Then X, A_3 are on opposite ends of a 1D very set, so they have equal power. Then $P(X) = P(A_1) = P(A_2)$ contradicting the fact the endpoints have the unique highest power. Therefore, since all sides only have 2 points on them, and all vertices have equal power, all sides are equal. □

Lemma 3. *H is a regular polygon.*

Proof. Let's look at the segment A_1A_3. Say that on it we have a very set of size $k-1$. By uniqueness and the construction in (a), and the fact that $P(A_1) = P(A_2) = P(A_3)$, we get that A_1, A_2, A_3 are 3 vertices of a regular k-gon. Now the very set on segment A_1A_3 under inversion at A_2 would map to a regular k-gon. So all vertices of this regular k-gon would be in our set. Assuming that not all angles are equal taking the largest angle who is adjacent to a smaller angle, we contradict convexity. So all angles are equal. Combining this with Lemma 1, H is a regular polygon. □

Lemma 4. *H cannot have more than 4 vertices.*

Proof. Firstly, note that no points can be strictly any of the triangles $A_iA_{i+1}A_{i+2}$. (*) Or else, inverting with center A_{i+1} we get a point outside H. First, let's do if m (number of vertices) is odd. Let $m = 2k+1$. $(k \geq 2)$ Look at the inversive image of A_{2k+1} under inversion with center A_2. Say it maps to X. Note that $P(X) < P(A_i)$ for any i. Now look at the line $A_{k+2}X$. Since A_{k+2} is an endpoint, but $P(X) < P(A_{k+2})$, the other endpoint of this 1D very set must be on ray $A_{k+2}X$ past X, contradicting (*), since no other vertices of H are on this ray. Similarly for m even and ≥ 6 we can also find 2 points like these who contain no other vertices in H on the line through them. □

Lemma 5. *We only have 2 distinct very sets in 2D (up to scaling), an equilateral triangle (when $n = 3$) and a square with its center (when $n = 5$).*

Proof. First if H has 3 points, then by (*) in Lemma 3, no other points can lie inside H. So we get an equilateral triangle. If H has 4 points, then by (*) in Lemma 3, the only other point that we can add into our set is the center of the square. This also must be added, and this gives a very set of size 5. □

Hence, the maximal size is 5. ■

This problem was proposed by Sammy Luo. This solution was given by Yang Liu.

4. Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be real numbers strictly between 0 and 1. For any subset S of $\{1, 2, \ldots, n\}$, define

$$f(S) = \prod_{i \in S} a_i \cdot \prod_{j \notin S} (1 - a_j).$$

Suppose that $\sum_{|S| \text{ odd}} f(S) = \frac{1}{2}$. Prove that $a_k = \frac{1}{2}$ for some k. (Here the sum ranges over all subsets of $\{1, 2, \ldots, n\}$ with an odd number of elements.)

Proposed by Kevin Sun.

Solution. Let $X = \sum_{|S| \text{ odd}} f(S)$. Consider n unfair coins which shows heads with probabilities $a_1, a_2, \ldots, a_n$. Observe that X computes the probability that an odd number of heads is obtained. Thus, it is clear that if $a_k = \frac{1}{2}$ for some k, then $X = \frac{1}{2}$.

Consequently $X - \frac{1}{2}$ is divisible by the polynomial $\prod_{i=1}^{n} \left(a_i - \frac{1}{2}\right)$. Since both are degree n, they must be equal up to scaling. Thus the conclusion follows. ■

This problem and solution were proposed by Kevin Sun.

5. Let ABC be a triangle with circumcenter O and orthocenter H. Let ω_1 and ω_2 denote the circumcircles of triangles BOC and BHC, respectively. Suppose the circle with diameter $\overline{AO}$ intersects ω_1 again at M, and line AM intersects ω_1 again at X. Similarly, suppose the circle with diameter $\overline{AH}$ intersects ω_2 again at N, and line AN intersects ω_2 again at Y. Prove that lines MN and XY are parallel.

Proposed by Sammy Luo.

Remark. Originally, the problem was phrased with respect to arbitrary isogonal conjugates in place of O and H. The modified version admits additional properties. In this version, X is the intersection of the tangents at B and C, while Y is the reflection of A across the midpoint of $\overline{BC}$.

Solution 1.

Since $\angle PMX = \angle QNY = \frac{\pi}{2}$, we derive

$$\angle PBX = \angle QBY = \angle PCX = \angle QCY = \frac{\pi}{2}.$$

Thus

$$\angle ABY = \frac{\pi}{2} + \angle ABQ = \angle PBC + \frac{\pi}{2} = \pi - \angle CBX,$$

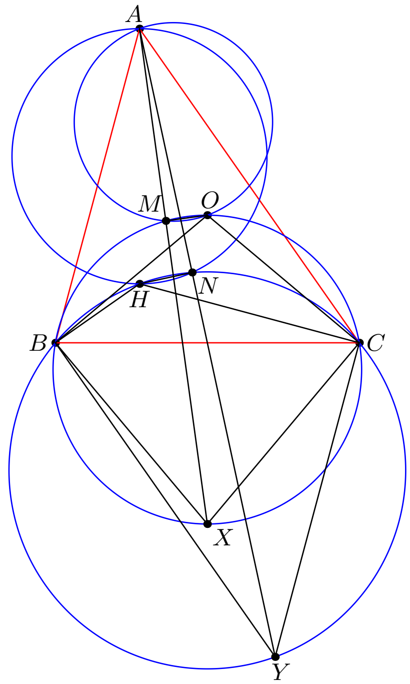
so X and Y are isogonal with respect to $\angle B$. However, similar angle chasing gives that they are isogonal with respect to $\angle C$. Thus they are isogonal conjugates with respect to ABC. (In particular, $\angle BAY = \angle XAC$.)

Also, $\angle ABY = \pi - \angle CBX = \pi - \angle CMX = \angle AMC$; hence $\triangle ABY \sim \triangle AMC$. Similarly, $\triangle ABN \sim \triangle AXC$. Thus $\frac{AN}{AB} = \frac{AC}{AX}$, and $\frac{AB}{AY} = \frac{AM}{AC}$. Multiplying, we get that $\frac{AN}{AY} = \frac{AM}{AX}$ which implies the conclusion. ■

This first solution was suggested by Kevin Sun.

Remark. The points M and N are also isogonal conjugates.

Solution 2. We apply barycentric coordinates with respect to triangle ABC (and as usual we apply Conway's Notation). Remark that the circle with diameter $\overline{AO}$ is the circumcircle of $A = (1, 0, 0)$ and the midpoints $M_B = (1 : 0 : 1)$ and $M_C = (1 : 1 : 0)$. Similarly, the circle with diameter $\overline{AH}$ is the circumcircle of $A = (1, 0, 0)$ and the feet of the altitudes $K_B = (S_C : 0 : S_A)$ and $K_C = (S_B : S_A : 0)$. It is then straightforward to derive the following



equations (using the standard formulas $2(S_{AB} + S_{BC} + S_{CA}) = a^2 S_A + b^2 S_B + c^2 S_C = 16K^2$, where K is the area of ABC.)

$$\begin{aligned}
(AM_BM_C) &: a^2yz + b^zx + c^2xy = (x+y+z)\left(\frac{1}{2}c^2y + \frac{1}{2}b^2z\right) \\
(AK_BK_C) &: a^2yz + b^zx + c^2xy = (x+y+z)\left(S_Bc^2y + S_Cz\right) \\
(BOC) &: a^2yz + b^zx + c^2xy = (x+y+z)\left(\frac{b^2c^2}{2S_A}x\right) \\
(BHC) &: a^2yz + b^zx + c^2xy = (x+y+z)\left(2S_ax\right)
\end{aligned}$$

It is now straightforward to check $M = (2S_A : b^2 : c^2)$ and $N = (a^2 : 2S_A : 2S_A)$ are the coordinates of M and N (by checking that they lie on the respective required circles). Therefore $\overline{AM}$ is a symmedian, whence it is clear that the intersection of the two tangents $X = (-a^2 : b^2 : c^2)$ is the correct form for X (one can also verify directly that this lies on (BOC)). Analogously we find $Y = (-1 : 1 : 1)$ follows from $\overline{AN}$ being a median (and again this can also be verified using coordinates only).

It remains to prove that $\overline{MN}$ and $\overline{XY}$ are parallel. By normalizing and comparing the x-coordinates, we find that

$$\frac{AM}{AX} = \frac{1 - \frac{2S_A}{2S_A+b^2+c^2}}{1 - \frac{-a^2}{-a^2+b^2+c^2}} = \frac{-a^2+b^2+c^2}{-a^2+2b^2+2c^2}$$

and

$$\frac{AN}{NX} = \frac{1 - \frac{a^2}{a^2+4S_A}}{1-(-1)} = \frac{2S_A}{a^2+4S_A} = \frac{-a^2+b^2+c^2}{-a^2+2b^2+2c^2}$$

and we are done. ■

This second solution was suggested by Sam Korsky.

Remark. This solution is clearly back-constructed. If the points (and hence coordinates of) X and Y are predicted from a well-drawn diagram, then one can use single linear computations to obtain the points M and N (as opposed to quadratics). Simply parametrize M as $(t : b^2 : c^2)$ and then consider the radical axis of (AOM) and (BOC), obtained by merely subtracting the two circle's equations.

Solution 3. First, remark that $\overline{OX}$ is a diameter of (BOC), meaning X is the intersection of the tangents to (ABC) at B and C. In particular $\overline{AX}$ is a symmedian. Next, notice that $\overline{HY}$ is a diameter of (BHC), meaning Y is the reflection of A over the midpoint of $\overline{BC}$. In particular $\overline{AX}$ is a median.

Now we claim that (AMB) and (AMC) are tangent to AC and AB, respectively. This follows from angle chasing via

$$\angle ABM = \angle B - \angle MBC = \angle B - \angle MXC = \cdots = \angle MAC.$$

Similarly, we claim that (ANB) and (ANC) are both tangent to BC. This just follows from

$$\angle BAN = \angle NYC = \angle NBC.$$

Now invert at A with radius $\sqrt{AB \cdot AC}$ and then reflect around the angle bisector of A. This map sends B to C. Using the tangencies above, we see that M is mapped to Y and N is mapped to X, so $AM \cdot AX = AN \cdot AY = AB \cdot AC$ and the conclusion follows. ■

This third solution was suggested by Michael Ren.

This problem was proposed by Sammy Luo.

6. A $2^{2014}+1$ by $2^{2014}+1$ grid has some black squares filled. The filled black squares form one or more snakes on the plane, each of whose heads splits at some points but never comes back together. In other words, for every positive integer n greater than 2, there do not exist pairwise distinct black squares $s_1, s_2, \ldots, s_n$ such that s_i and s_{i+1} share an edge for $i = 1, 2, \ldots, n$ (here $s_{n+1} = s_1$).

   What is the maximum possible number of filled black squares?

   Proposed by David Yang.

Answer. If $n = 2^m + 1$ is the dimension of the grid, the answer is $\frac{2}{3}n(n+1) - 1$. In this particular instance, $m = 2014$ and $n = 2^{2014} + 1$.

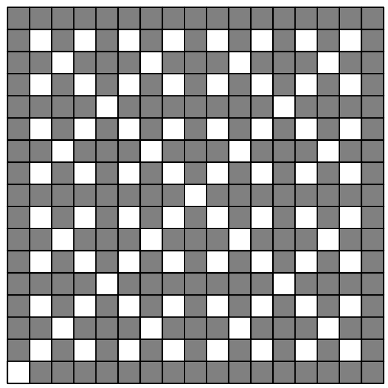
Solution 1. Let $n = 2^m + 1$. Double-counting square edges yields $3v + 1 \le 4v - e \le 2n(n+1)$, so because $n \not\equiv 1 \pmod 3$, $v \le 2n(n+1)/3 - 1$. Observe that if $3 \nmid n - 1$, equality is achieved iff (a) the graph formed by black squares is a connected forest (i.e. a tree) and (b) all but two square edges belong to at least one black square.

We prove by induction on $m \ge 1$ that equality can in fact be achieved. For $m = 1$, take an "H-shape" (so if we set the center at $(0,0)$ in the coordinate plane, everything but $(0, \pm 1)$ is black); call this G_1. To go from G_m to G_{m+1}, fill in $(2x, 2y)$ in G_{m+1} iff (x, y) is filled in G_m, and fill in (x, y) with x, y not both even iff $x + y$ is odd (so iff one of x, y is odd and the other is even). Each "newly-created" white square has both coordinates odd, and thus borders 4 (newly-created) black squares. In particular, there are no new white squares on the border (we only have the original two from G_1). Furthermore, no two white squares share an edge in G_{m+1}, since no square with odd coordinate sum is white. Thus G_{m+1} satisfies (b). To check that (a) holds, first we show that $(2x_1, 2y_1)$ and $(2x_2, 2y_2)$ are connected in G_{m+1} iff (x_1, y_1) and (x_2, y_2) are black squares (and thus connected) in G_m (the new black squares are essentially just "bridges"). Indeed, every path in G_{m+1} alternates between coordinates with odd and even sum, or equivalently, new and old black squares. But two black squares (x_1, y_1)

and (x_2, y_2) are adjacent in G_m iff $(x_1 + x_2, y_1 + y_2)$ is black and adjacent to $(2x_1, 2y_1)$ and $(2x_2, 2y_2)$ in G_{m+1}, whence the claim readily follows. The rest is clear: the set of old black squares must remain connected in G_{m+1}, and all new black squares (including those on the boundary) border at least one (old) black square (or else G_m would not satisfy (b)), so G_{m+1} is fully connected. On the other hand, G_{m+1} cannot have any cycles, or else we would get a cycle in G_m by removing the new black squares from a cycle in G_{m+1} (as every other square in a cycle would have to have odd coordinate sum). ■

This problem and solution were proposed by David Yang.

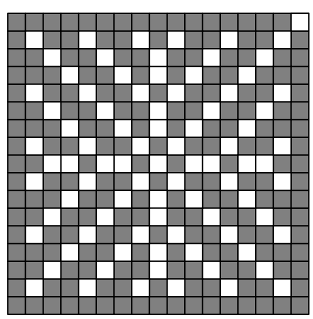
Solution 2. As above, we can show that there are at most $\frac{2}{3}n(n+1) - 1$ black squares. We provide a different construction now for $n = 2^k + 1$.



Consider the grid as a coordinate plane (x, y) where $0 \le x, y \le 2^m$. Color white the any square (x, y) for which there exists a positive integer k with $x \equiv y \equiv 2^{k-1} \pmod 2)^k$. Then, color white the square $(0, 0)$. Color the remaining squares black. Some calculations show that this is a valid construction which achieves $\frac{2}{3}n(n+1) - 1$. ■

This second solution was suggested by Kevin Sun.

Solution 3. We can achieve the bound of $\frac{2}{3}n(n+1) - 1$ as above. We will now give a construction which works for all $n = 6k + 5$. Let $M = 3k + 2$.



Consider the board as points (x, y) where $-M \le x, y \le M$. Paint white the following types of squares:

- The origin $(0, 0)$ and the corner (M, M).
- Squares of the form $(\pm a, 0)$ and $(0, \pm a)$, where $a \not\equiv 1 \pmod 3$ and $0 < a < M$.
- Any square $(\pm x, \pm y)$ such that $y - x \equiv 0 \pmod 3$ and $0 < x, y < M$.

Paint black the remaining squares. This yields the desired construction. ■

This third solution was suggested by Ashwin Sah.