15^{th} Everyone Lives at Most Once

Lincoln, Nebraska

Day I 8:00 AM - 12:30 PM

June 15, 2013

1. Let $a_1, a_2, \ldots, a_9$ be nine real numbers, not necessarily distinct, with average m. Let A denote the number of triples $1 \le i < j < k \le 9$ for which $a_i + a_j + a_k \ge 3m$. What is the minimum possible value of A?

2. Let a, b, c be positive reals satisfying $a + b + c = \sqrt[7]{a} + \sqrt[7]{b} + \sqrt[7]{c}$. Prove that $a^a b^b c^c \ge 1$.

3. Let $m_1, m_2, \ldots, m_{2013} > 1$ be 2013 pairwise relatively prime positive integers and $A_1, A_2, \ldots, A_{2013}$ be 2013 (possibly empty) sets with $A_i \subseteq \{1, 2, \ldots, m_i - 1\}$ for $i = 1, 2, \ldots, 2013$. Prove that there is a positive integer N such that

$$N \le (2\,|A_1| + 1)\,(2\,|A_2| + 1) \cdots (2\,|A_{2013}| + 1)$$

and for each $i = 1, 2, \ldots, 2013$, there does *not* exist $a \in A_i$ such that m_i divides $N - a$.

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Day II 8:00 AM - 12:30 PM

June 16, 2013

4. Triangle ABC is inscribed in circle ω. A circle with chord BC intersects segments AB and AC again at S and R, respectively. Segments BR and CS meet at L, and rays LR and LS intersect ω at D and E, respectively. The internal angle bisector of $\angle BDE$ meets line ER at K. Prove that if $BE = BR$, then $\angle ELK = \frac{1}{2}\angle BCD$.

5. For what polynomials $P(n)$ with integer coefficients can a positive integer be assigned to every lattice point in $\mathbb{R}^3$ so that for every integer $n \geq 1$, the sum of the n^3 integers assigned to any $n \times n \times n$ grid of lattice points is divisible by $P(n)$?

6. Consider a function $f : \mathbb{Z} \to \mathbb{Z}$ such that for every integer $n \geq 0$, there are at most $0.001n^2$ pairs of integers (x, y) for which $f(x + y) \neq f(x) + f(y)$ and $\max\{|x|, |y|\} \leq n$. Is it possible that for some integer $n \geq 0$, there are more than n integers a such that $f(a) \neq a \cdot f(1)$ and $|a| \leq n$?