English Language Master's Open

Day I 8:00 AM – 12:30 PM

June 18, 2011

Write your number and team abbreviation, but not your name, on top of all pages turned in.

1. Let $ABCD$ be a convex quadralateral. Let E, F, G, H be points on segments AB, BC, CD, DA, respectively, and let P be intersection of EG and FH. Given that quadrilaterals $HAEP, EBFP, FCGP, GDHP$ all have inscribed circles, prove that $ABCD$ also has an inscribed circle.

2. Wanda the Worm likes to eat Pascal's triangle. One day, she starts at the top of the triangle and eats $\binom{0}{0} = 1$. Each move, she travels to an adjacent positive integer and eats it, but she can never return to a spot that she has previously eaten. If Wanda can never eat numbers a, b, c such that $a + b = c$, proof that it is possible for her to eat 100,000 numbers in the first 2011 rows given that she is not restricted to traveling only in the first 2011 rows.

   (Here, the $n+1^{\text{st}}$ row of Pascal's triangle consists of entries of the form $\binom{n}{k}$ for integers $0 \leq k \leq n$. Thus, the entry $\binom{n}{k}$ is considered adjacent to the entries $\binom{n-1}{k-1}$, $\binom{n-1}{k}$, $\binom{n}{k-1}$, $\binom{n}{k+1}$, $\binom{n+1}{k}$, $\binom{n+1}{k+1}$.)

3. Determine whether there exists a sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers such that the following holds:

   - For all $n \geq 0$, $a_n \neq 0$.
   - There exist real numbers x and y such that $a_{n+2} = xa_{n+1} + ya_n$ for all $n \geq 0$.
   - For all positive real numbers r, there exists positive integers i and j such that $|a_i| < r < |a_j|$.

Note: Our Engrish level beginner. Please excuse us any typos and us help fix mistake.

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4. Find all functions $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the positive reals, such that whenever $a > b > c > d > 0$ are reel numbers with $ad = bc$,

$$f(a + d) + f(b - c) = f(a - d) + f(b + c).$$

5. Let $p > 13$ be a prime of the the form $2q+1$, where q is prime. Find the number of ordered pairs of integers (m, n) such that $0 \leq m < n < p - 1$ and

$$3^m + (-12)^m \equiv 3^n + (-12)^n \pmod{p}.$$

6. Consider the infinite grid of lattice points in $\mathbb{Z}^3$. Little D and Big Z play a game, where Little D first loses a shoe on an unmunched point in the grid. Then, Big Z munches a shoe-free plane perpendicular to one of the coordinate axes. They continue to alternature turns in this fashion, with Little D's goal to loose a shoe on each of n consecutive lattice points on a line parrallel to one of the coordinate axes. Determine all n for which Little D can accomplish his goal.

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