Exceedingly Luck-based Math Olympiad Solutions

1. Determine all (not necessarily finite) sets S of points in the plane such that given any four distinct points in S, there is a circle passing through all four or a line passing through some three.

Solution The answer is any subset of a fixed circle, any subset of a fixed line, any subset of a fixed line with one additional point not on the line, or four points on a circle, with a fifth point as the intersection of its diagonals or the intersection of a pair of its sides (outside the circle). It is clear that these sets all satisfy the needed condition.

First, assume that some four points on S lie on a circle, say A, B, C, and D, in that order. We claim that the rest of S lies on the circle, or S consists of exactly one more point, either the intersection of the diagonals of the quadrilateral formed by A, B, C, D, or the intersection of two sides of the quadrilateral outside the circle. Assume there exists a point E in S, not on the circle. Then, E, A, B, C are not concyclic, and A, B, C are not collinear, so E lies on one of segments AB, BC, CA. Without loss of generality, say E lies on AB. Now, consider E, B, C, D; by similar logic to before, E lies on BC, CD, or DB, but since E, A, B are collinear, and A, B, C are not collinear, we need E, B, D to be collinear, that is, $E = AB \cap CD$.

However, note that at most one of these intersection points can be in S, because if not, it is easy to check that we will get a triangle with a point in the interior in S, in which we have four points that cannot satisfy the given condition. Additionally, we can have at most four points on the circle, because if we have five, say A, B, C, D, E, and a sixth point P in S lies off the circle (we know that at most one such point exists, from before), then it must be the intersection of two lines formed by A, B, C, D; without loss of generality, say $P = AB \cap CD$. Also, it must be the intersection of two lines formed by A, B, C, E. But $P \in AB$, so $P \in CE$, which is impossible, since this means C, D, E are collinear.

We are now left with the case when no four points are concyclic, which means that any four points in S have some three collinear. Starting with four points A, B, C, D, some three are collinear, say A, B, C. But for any other point $E \in S$, some three of A, B, C, E are collinear, meaning all four must be collinear. Thus, all or all but one of our points must lie on the same line.

This exhausts all cases, and when there are fewer than four points in S, the statement is vacuously true. It follows that the only possible sets S are those described above.

2. Let r and s be positive integers. Define $a_0 = 0$, $a_1 = 1$, and $a_n = ra_{n-1} + sa_{n-2}$ for $n \geq 2$. Let $f_n = a_1 a_2 \cdots a_n$. Prove that $\dfrac{f_n}{f_k f_{n-k}}$ is an integer for all integers n and k such that $0 < k < n$.

Solution Lemma: For nonnegative integers x, y, $a_{x+y} = a_x a_{y+1} + sa_{x-1}a_y$. We will prove this by induction on y. We have two base cases, $y = 0$ and $y = 1$. When $y = 0$ we simply need to prove that $a_x = a_x$, which is trivial. When $y = 1$, we need to prove that $a_{x+1} = a_x a_1 + sa_{x-1}$. But $a_1 = r$, so this is true directly from the recurrence relation. Now suppose we know that $a_{x+y} = a_x a_{y+1} + sa_{x-1}a_y$ for $y = k$ and $y = k + 1$. Then we have $a_{x+k+2} = ra_{x+k} + sa_{x+k+1} = ra_x a_{k+1} + sa_x a_{k+2} + rsa_{x-1}a_k + s^2 a_{x-1}a_{k+1} = a_x a_{k+3} + a_{x-1}a_{k+2}$, which is exactly what we want to show for $y = k + 2$. This completes our induction.

Now for the main proof, let $f_0 = 1$. Then we will prove the claim by induction on n. The base cases, $n = 0$ or $k = 0$, are trivial. Suppose we know that $\frac{f_n}{f_k f_{n-k}}$ is an integer for all smaller n. Then we have $\frac{f_n}{f_k f_{n-k}} = \frac{f_{n-1}a_{n-k+k}}{f_k f_{n-k}} = \frac{f_{n-1}(a_{n-k}a_{k+1}+sa_{n-k-1}a_k)}{f_k f_{n-k}} = \frac{f_{n-1}a_{n-k}a_{k+1}}{f_k f_{n-k}} + \frac{f_{n-1}sa_{n-k-1}a_k}{f_k f_{n-k}} = \frac{f_{n-1}}{f_k f_{n-k-1}} \cdot a_{k+1} + \frac{f_{n-1}}{f_{k-1} f_{n-k}} \cdot sa_{n-k-1}$, which is clearly an integer by the inductive hypothesis. This completes the induction and the proof.

3. Let $n > 1$ be a positive integer. A 2-dimensional grid, infinite in all directions, is given. Each 1 by 1 square in a given n by n square has a counter on it. A *move* consists of taking n adjacent counters in a row or column and sliding them each by one space along that row or column. A *returning sequence* is a finite sequence of moves such that all counters again fill the original n by n square at the end of the sequence.

   (a) Assume that all counters are distinguishable except two, which are indistinguishable from each other. Prove that any distinguishable arrangement of counters in the n by n square can be reached by a returning sequence.

(b) Assume all counters are distinguishable. Prove that there is no returning sequence that switches two counters and returns the rest to their original positions.

Solution (a) First, we will find a way to 3-cycle some counters, and then use these cycles to construct any board.

Lemma 1. It is possible to cycle any three adjacent counters in an L-formation, while leaving all other counters unchanged.

Proof. Suppose we have counters c_1, c_2, and c_3 in such a formation. Suppose without loss of generality that c_1 is directly above c_2 and that c_3 is directly to the right of c_2. Make the following four moves:

i. Slide the column containing c_1 and c_2 down.
ii. Slide the row now containing c_1 and c_3 right.
iii. Slide the column now containing c_2 and c_3 up.
iv. Slide the row now containing c_1 and c_2 right.

This cycles the three counters. Note that performing this cycle twice is simply cycling the in the other direction.

Now we can use this cycle to get any grid we want. To show this, we think of this as starting from a given grid, from where we aim to get back to the original position. To show that this can be done, we do induction on n.

Base Case. $n = 2$. First, we do a cycle, if necessary, to get the correct counter into the top-left position. Then, we do another cycle, consisiting of the other three squares, to get the correct counter into the top-right position. Then we are done, because the remaining two counters are indistinguishable and thus will be correctly placed.

Inductive Step. Assume that such an algorithm is possible for an $(n-1) \times (n-1)$ board. In our $n \times n$ board, we can use these cycles to get the correct counters into the topmost row, one-by-one. We then finish the remaining positions in the leftmost column. We are now left with an $(n-1) \times (n-1)$ board, so we apply the inductive hypothesis to finish.

(b) First, I claim that any returning sequence must use an even number of moves. To see this, consider all of the positions that contain a counter, and let S be the sum of all the x-coordinates and y-coordinates of these positions. Any move will add either 1 or -1 to n of the x-coordinates or y-coordinates, thus changing S by n. If we look at S mod $2n$, this is equivalent to always adding n to S. In a returning sequence, S must be the same as it was originally, so there must be an even number of moves to make S agree with its original value mod $2n$.

Now, instead of thinking of these counters as being on an infinite grid, we only look at their coordinates mod n. Any valid move will simply cycle the coordinates (either x or y) mod n. Then, at any point, for any position (x, y), there will be exactly one counter that has those coordinates mod n, so each move is simply an n-cycle of these mod n coordinates. Since any returning sequence consists of an even number of moves, the coordinates will ultimately go through an even number of n cycles, and the composition of these cycles will be an even permuation. However, the transposition of any two counters is an odd permuation, so a returning sequence that switches only two counters is impossible.

4. Determine all strictly increasing functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $nf(f(n)) = f(n)^2$ for all positive integers n.

Solution The answer is $f(n) = n$ for all $n = 1, 2, \ldots, N$ for some positive integer N, and $f(n) = an$ for fixed positive integer a for $n > N$. It is not difficult to check that all of these f work, since if $n \leq N$, $nf(f(n)) = n^2 = f(n)^2$, and if $n > N$, $nf(f(n)) = a^2n^2 = f(n)^2$.

First, say $f(n) = an$ for some positive integer n, such that $an \in \mathbb{N}$. Then, $nf(f(n)) = nf(an) = f(n)^2 = a^2n^2$, so $f(an) = a(an)$. It follows easily by induction that for all non-negative integers k, $f(a^kn) = a^{k+1}n$. In particular, $an, a^2n, \ldots$ are all integers, which implies that a itself is an integer, since if a prime p divides the denominator of a, when a is raised to a large enough power, the power of p can no longer divide n, making a^kn non-integral for large enough k.

Now, assume that $f(n_1) = an_1$ and $f(n_2) = bn_2$ for some distinct positive integers $a, b > 1$. Without loss of generality, say $a < b$. Choose a positive integer k such that $a^kn_1 > n_2$. Then, we have

$f(a^k n_1) = a^{k+1}n_1$, and $f(n_2) = bn_2$, so that $a^{k+1}n_1 > bn_2$, as f is strictly increasing. Applying f repeatedly to both sides, we find that $a^{k+e}n_1 > b^e n_2$ for all integers $e > 0$, but this is impossible for large enough e, as $b > a$. Thus, we must have $a = b$.

Thus, for some positive integer a, for all n, either $f(n) = n$ or $f(n) = an$. Let n be an integer such that $f(n) = an$, and $a > 1$. Then, assume we have some $m > n$ such that $f(m) = m$. For the unique k such that $a^k n \leq m < a^{k+1}n$, note that $f(a^k n) = a^{k+1}n$. But since $m \geq a^k n$, as f is increasing, we need $f(m) = m \geq a^{k+1}n$, a contradiction. It follows that either $f(n) = an$ for all n, or there exists a positive integer N such that $f(n) = n$ for all $n \leq N$ and $f(n) = an$ for $n > N$, as claimed.

5. 2010 MOPpers are assigned numbers 1 through 2010. Each one is given a red slip and a blue slip of paper. Two positive integers, A and B, each less than or equal to 2010 are chosen. On the red slip of paper, each MOPper writes the remainder when the product of A and his or her number is divided by 2011. On the blue slip of paper, he or she writes the remainder when the product of B and his or her number is divided by 2011. The MOPpers may then perform either of the following two operations:

   - Each MOPper gives his or her red slip to the MOPper whose number is written on his or her blue slip.
   - Each MOPper gives his or her blue slip to the MOPper whose number is written on his or her red slip.

   Show that it is always possible to perform some number of these operations such that each MOPper is holding a red slip with his or her number written on it.

Solution Note that 2011 is prime, so each slip of paper of a given color has a different number on it. All arithmetic from now on will be done modulo 2011 unless otherwise stated. Now suppose that person i has red slip Ai and blue slip Bi. Then person $B^{-1}i$ has blue slip i, so after performing the first operation, person i will have red slip $AB^{-1}i$ and still have blue slip Bi. Similarly, if the second operation were performed instead, then person i would have red slip Ai and blue slip $A^{-1}Bi$. This holds for every index i, so we can represent the operations simply as $(A, B) \to (AB^{-1}, B)$ and $(A, B) \to (A, A^{-1}B)$.

Now consider a primitive root g modulo 2011 and write $A = g^a$ and $B = g^b$ for some natural numbers a, b. Then, now considering arithmetic in natural numbers, we can write the operations as $(a, b) \to (a - b, b)$ and $(a, b) \to (a, b - a)$. These two operations allow us to apply the Euclidean algorithm to reduce one of these two values to 0. If a becomes 0, every MOPper has his or her red slip, and so we are done. If b becomes 0, then we notice that the second to last pair must have been (a, a), in which case we can simply go to $(0, a)$ instead. However, if we started at $(a, 0)$ then we cannot do this, so we apply the second operation repeatedly. We notice that as the multiples of a are cyclic modulo 2010 and these values are exponents of a primitive root, eventually we will reach a pair equivalent to (a, a), at which point we can perform the first operation to arrive at $(0, a)$, as desired.

6. Let ABC be a triangle with circumcircle ω, incenter I, and A-excenter I_A. Let the incircle and the A-excircle hit BC at D and E, respectively, and let M be the midpoint of arc BC without A. Consider the circle tangent to BC at D and arc BAC at T. If TI intersects ω again at S, prove that SI_A and ME meet on ω.

Solution Note that the homothety around T taking the small circle to ω. This homothety takes D to M as the tangents are parallel, so T, D, M are collinear. Then note that $\angle MBD = \frac{1}{2} \overset{\frown}{MC} = \frac{1}{2} \overset{\frown}{MB} = \angle MTB$, so $\triangle MBD \sim \triangle MTB$, so $MD \cdot MT = MB^2$. Let ME intersect ω at R. Then it suffices to show that R, S, I_A are collinear. Note that $MB = MI_A = MI = MC$. Additionally, notice that E and R are the reflections across the perpendicular bisector of BC of D and T, respectively. Therefore, $MD = ME$ and $MT = MR$, so $MI_A^2 = ME \cdot MR$, so $\triangle MEI_A \sim \triangle MI_AR$ and so $\angle MI_AE = \angle MRI_A$. Additionally, as $I_AE \perp BC$, we have $I_AE \parallel ID$, so $\angle MI_AE = \angle MID$. Finally, $MI^2 = MD \cdot MT$, so $\angle MID = \angle MTI = \angle MRS$ because $MTRS$ is cyclic. Therefore, $\angle MRI_A = \angle MRS$, so R, S, I_A are collinear as desired.